Inner Product Spaces: Orthogonal Complements

Recall: Let W be a subspace of an inner product space V. Then

$$W^{\perp} = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \text{ for all } \mathbf{w} \in W\}.$$

Example: $\mathcal{B} = \left\{1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x\right\}$ is an orthogonal basis for $\mathcal{P}_3(\mathbb{R})$ with respect to the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

So, if

$$W = \operatorname{Span}\left(\left\{1, \frac{3}{2}x^2 - \frac{1}{2}\right\}\right)$$

Useful Facts From Tutorial Worksheet 10: Let W be a subspace of the inner product space V. Then

1. $W^{\perp}$ is a subspace of V
2. $W \cap W^{\perp} = \{\mathbf{0}\}$
3. If $\mathbf{x} \in V$ then we can write

Theorem: Let W be a subspace of the inner product space V. We have

$$V = W \oplus W^{\perp}.$$

Corollary: With V and W as above, if V is finite-dimensional then

$$\dim V = \dim W + \dim W^{\perp}.$$

Application – Method Of Least Squares

Set-Up: Suppose we have collected data and want to model it by an equation of the form $y = a + bx + cx^2$.

Call the data points $(x_1, y_1), \ldots, (x_n, y_n)$.

Use $\mathbb{R}^n$ with respect to the dot product! For the sake of notation, we let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \mathbf{x^2} = \begin{bmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{bmatrix}.$$

So we want to minimize

$$||\mathbf{y} - (a\mathbf{1} + b\mathbf{x} + c\mathbf{x^2})||^2.$$

To find a, b, c we need to find $\mathbf{v} \in \text{Span}(\{\mathbf{1}, \mathbf{x}, \mathbf{x^2}\})$ closest to $\mathbf{y}$.

That is,

Problem: $\{\mathbf{1}, \mathbf{x}, \mathbf{x^2}\}$ is

Fix: Let W be a subspace of an inner product space V and let $\mathbf{v} \in V$. Then we have the following lemma:

Lemma: With W and V as above,

1. $\mathbf{w} = \text{proj}_W(\mathbf{v})$ if and only if

2. Let $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ be *any* basis for W. Then $\mathbf{v} \in W^\perp$ if and only if

So ... back to our problem ... we need to find $a, b, c \in \mathbb{R}$ such that

$$(\mathbf{y} - (a\mathbf{1} + b\mathbf{x} + c\mathbf{x^2})) \cdot \mathbf{1} = 0$$
$$(\mathbf{y} - (a\mathbf{1} + b\mathbf{x} + c\mathbf{x^2})) \cdot \mathbf{x} = 0$$
$$(\mathbf{y} - (a\mathbf{1} + b\mathbf{x} + c\mathbf{x^2})) \cdot \mathbf{x^2} = 0$$

Re-organization: Let

$$\mathbf{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \mathbf{1} & \mathbf{x} & \mathbf{x^2} \end{bmatrix}.$$

Example: Suppose we have the data

x	-1	0	1	2
y	4	1	1	-1

In general ... suppose we are given the data points

x	x_1	x_2	$\cdots$	x_n
y	y_1	y_2	$\cdots$	y_n

If you want to find an equation $y = a_0 + a_1 x + \cdots + a_k x^k$ of best fit, let

$$\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{x^2} = \begin{bmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{bmatrix}, \ldots, \mathbf{x^k} = \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix}.$$

Letting

$$X = \begin{bmatrix} \mathbf{1} & \mathbf{x} & \mathbf{x^2} & \cdots & \mathbf{x^k} \end{bmatrix}$$

gives

$$\mathbf{a} = (X^T X)^{-1} X^T \mathbf{y}$$

as the equation of best fit!

Question: