Inner Product Spaces

Motivation: Recall that in $\mathbb{R}^2$ we have length of vectors and angles between vectors: in particular, if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

are in $\mathbb{R}^2$, then

- the *length* of $\mathbf{x}$ is

- the Cosine Rule says that if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ then

Question: How do we deal with length and angles in $\mathbb{R}^n$? $\mathcal{P}_2(\mathbb{R})$? $M_{2\times 2}(\mathbb{C})$?

Definition: Let V be a vector space over the field $\mathbb{F}$. An **inner product on** V is a function

$$\langle \quad , \quad \rangle : V \times V \to \mathbb{F}$$

such that the following properties hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all α in $\mathbb{F}$:

1. $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$
2. $\langle \alpha\mathbf{v}, \mathbf{w} \rangle =$
3. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle =$
4.

A vector space V equipped with an inner product is called an **inner product space**.

Examples:

1. Let $V = \mathbb{C}^n, \mathbb{F} = \mathbb{C}$. Define the **standard Hermitian inner product** as

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 \overline{w_1} + \cdots + v_n \overline{w_n}.$$

   This is an inner product!

2. Let $V = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$. Define

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

3. Let $V = \mathcal{P}_3(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$. Define

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x) q(x)\, dx.$$

4. Let $V = \mathcal{P}_2(\mathbb{C})$ and $\mathbb{F} = \mathbb{C}$. Define

$$\langle p(x), q(x) \rangle = p(i)\overline{q(i)} + p(-i)\overline{q(-i)} + p(1)\overline{q(1)}.$$

This is an inner product!

5. Let V be any finite-dimensional vector space with basis $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$.

6. Let $V = \mathbb{C}^2$ and $\mathbb{F} = \mathbb{C}$. Define

$$\langle \mathbf{v}, \mathbf{w} \rangle = 2v_1\overline{w_1} - v_2\overline{w_2}.$$

We are now in position to define what we mean by the length of a vector in any inner product space.

Definition: Let V be an inner product space. The **length** (or **norm**) of a vector $\mathbf{v}$ in V is

Example: Using the inner product from Example 4 above:

We finish by stating some basic properties of inner products.

Proposition: Let V be an inner product space, $\mathbf{v}, \mathbf{w}, \mathbf{u} \in V$ and $\alpha \in \mathbb{F}$. We have

1. $||\mathbf{0}|| = 0$
2. $\langle \mathbf{0}, \mathbf{v} \rangle = 0$
3. $\langle \mathbf{v}, \alpha\mathbf{w} \rangle = \overline{\alpha}\langle \mathbf{v}, \mathbf{w} \rangle$
4. $\langle \mathbf{v}, \mathbf{u} + \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$