Linear Transformations: Images and Kernels

Goal: We now study two *very important* subspaces associated to a linear transformation.

Definition: Let $T : V \to W$ be a linear transformation.

1. The **kernel** (or **nullspace**) of T is the set

2. The **image** (or **range**) of T is the set

Proposition: Let $T : V \to W$ be a linear transformation. Then

1. $\ker(T)$ is a subspace of V;
2. $\operatorname{Im}(T)$ is a subspace of W.

Examples:

1. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}.$$

2. Define the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b+c) + (c-d)x^2.$$

Definition: Let V and W be vector spaces over the field $\mathbb{F}$ and let $T : V \to W$ be a linear transformation.

1. **Rank**$(T) = \dim(\mathrm{Im}(T))$.
2. **Nullity**$(T) = \dim(\ker(T))$.

Rank–Nullity Theorem: Let V and W be vector spaces over the field $\mathbb{F}$ with $\dim(V) = n$. Let $T : V \to W$ be a linear transformation. We have

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V) = n.$$

Example: Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^3$ be a linear transformation. Since $\text{Im}(T)$ is a subspace of $\mathbb{R}^3$, we have that

$$\dim(\text{Im}(T)) \leq \dim(\mathbb{R}^3) = 3.$$

We also know that $\dim(\mathcal{P}_3(\mathbb{R})) = 4$. Thus, by the Rank–Nullity Theorem,

Thus, although we don't know T, we do know that there is at least one *non-zero* polynomial $p(x) \in \mathcal{P}_3(\mathbb{R})$ such that $T(p(x)) = \mathbf{0}_{\mathbb{R}^3}$.