Linear Transformations: Extensions and Isomorphisms

Before moving on, let's look at another example of the use of the Rank–Nullity Theorem.

Example: Let $T : \mathbb{C}^4 \to M_{2\times 2}(\mathbb{C})$ be a linear transformation. We claim that $\ker(T) = \{\mathbf{0}_{\mathbb{C}^4}\}$ if and only if $\operatorname{Im}(T) = M_{2\times 2}(\mathbb{C})$.

A Quick Look At Fundamental Constructions

I - The first construction we will look at involves extensions.

Definition: Let V and W be vector spaces over the field $\mathbb{F}$. Let $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be an *ordered basis* for V and $\{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ be an *ordered set* of vectors in W. The function $T : V \to W$ defined as follows: if $\mathbf{v} = a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n}$ (where the a_j are unique scalars in $\mathbb{F}$), then

$$T(\mathbf{v}) = a_1\mathbf{w_1} + \cdots + a_n\mathbf{w_n}$$

(note: $T(\mathbf{v_j}) = \mathbf{w_j}$ for $j = 1, \ldots, n$) is called the **linear extension** of $T(\mathbf{v_j}) = \mathbf{w_j}$ for $j = 1, \ldots, n$.

Theorem: With the notation from the above definition, the linear extension of $T(\mathbf{v_j}) = \mathbf{w_j}$ for $j = 1, \ldots, n$ is a linear transformation.

Proposition: Let $\mathbf{w_1}, \ldots, \mathbf{w_n}$ be vectors in the vector space W. Let $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ be the standard basis for $\mathbb{R}^n$ and $T : \mathbb{R}^n \to W$ be the linear extension of $T(\mathbf{e_j}) = \mathbf{w_j}$ for $j = 1, \ldots, n$.

1. $\{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ is linearly independent if and only if

2. $Span(\{\mathbf{w_1}, \ldots, \mathbf{w_n}\}) = W$ if and only if

Example: Let $T : \mathbb{R}^3 \to \mathcal{P}_5(\mathbb{R})$ be the linear extension of

$$\begin{aligned} T((1,1,1)) &= x^2 + x^4 \\ T((1,1,0)) &= x + x^3 + x^5 \\ T((1,0,0)) &= 1 \end{aligned}$$

What is $T((0,0,1))$?

II - The second construction we consider involves the set of all linear transformations.

Definition: Let $S, T : V \to W$ be linear transformations between vectors spaces V and W over the field $\mathbb{F}$. Let $\alpha \in \mathbb{F}$.

1. $S + T : V \to W$ is defined by

2. $\alpha T : V \to W$ is defined by

Note: $S + T$ and αT are linear transformations.

Theorem: Let V and W be vector spaces over the field $\mathbb{F}$. Denote by $\mathcal{L}(V, W)$ the set of all linear transformations from V to W. Then the set $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$ under the two operations defined above.

Note:

Isomorphisms

Recall: $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$ "behave the same" even though they are different vector spaces. Let's make this comparison more formal.

Definition: A linear transformation $T : V \to W$ is said to be an **isomorphism** if there exists a linear transformation $S : W \to V$ such that

$$(S \circ T)(\mathbf{v}) = S(T(\mathbf{v})) = \mathbf{v}$$

and

$$(T \circ S)(\mathbf{w}) = T(S(\mathbf{w})) = \mathbf{w}$$

Example: Define the linear transformations

$$T : \mathbb{R}^3 \to \mathcal{P}_2(\mathbb{R})$$

by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a + bx + cx^2$$

and

$$S : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$$

by

$$S(e + fx + gx^2) = \begin{bmatrix} e \\ f \\ g \end{bmatrix}.$$

Question: How can we tell if a linear transformation $T : V \to W$ has an inverse $S : W \to V$?

We need a few preliminaries to answer this question.

Definition: Let $T : V \to W$ be a linear transformation between vector spaces.

1. T is called **injective** (or **one-to-one**) if

2. T is called **surjective** (or **onto**) if

Proposition: A linear transformation $T : V \to W$ is injective if and only if $\ker(T) = \{\mathbf{0_V}\}$.