Eigenvalues And Eigenvectors

Example: Let's look at an example from Tutorial Worksheets 5 & 6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{2(x+y+z)}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We use the two bases for $\mathbb{R}^3$:

- $\mathcal{B}$ is the standard basis
- $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Then

Question: How do we find the most insightful bases?

Let's focus on linear transformations $T : V \to V$. Ideally, we want our matrix representation to be a diagonal matrix. That is, we want a basis $\mathcal{D}$ for V so that

Question: Can we always find such a basis $\mathcal{D}$?

Definitions: Let $T : V \to V$ be a linear transformation and $A \in M_{n\times n}(\mathbb{F})$.

1. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists a *non-zero* vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of T associated to the eigenvalue λ.

2. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of A if

Proposition: Let $T : V \to V$ be a linear transformation and let λ be an eigenvalue of T. The eigenspace E_λ corresponding to λ is a subspace of V.

Examples:

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T((a, b, c)) = (a, 2b, -c).$$

   We have

2. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$. We have

3. Let $D : \mathcal{P}_4(\mathbb{C}) \to \mathcal{P}_4(\mathbb{C})$ be the differentiation transformation.

Question: How do we find eigenvectors and eigenvalues?

Answer: Let $A \in M_{n\times n}(\mathbb{F})$. We want to find $\mathbf{x} \neq \mathbf{0}$ such that

Example: Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \in M_{2\times 2}(\mathbb{F})$.

Note: $\det(A - \lambda I_n)$ is a polynomial in λ. It turns out that this special polynomial tells us tons about the matrix A and the linear transformation that it represents!

Definition: Let $A \in M_{n\times n}(\mathbb{F})$. The **characteristic polynomial of** A is $\Delta(\lambda) = \det(A - \lambda I_n)$.

Example: With the matrix A from the previous example, we have