Eigenvalues And Eigenvectors: Diagonalization

Recall: If $A \in M_{n\times n}(\mathbb{F})$ then the characteristic polynomial of A is $\Delta(\lambda) = \det(A - \lambda I_n)$ which is a polynomial in λ.

Proposition: Let $A \in M_{n\times n}(\mathbb{F})$. The eigenvalues of A are the roots of the characteristic polynomial of A (i.e., the values of λ such that $\det(A - \lambda I_n) = 0$).

Corollary: If $A \in M_{n\times n}(\mathbb{F})$ then $\det(A)$ is the product of its eigenvalues.

Proposition: Let $A \in M_{n\times n}(\mathbb{F})$ and let λ be an eigenvalue of A. The eigenspace E_λ corresponding to λ is $Null(A - \lambda I_n)$.

Example: Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Some Facts About Characteristic Polynomials

Some important facts about characteristic polynomials follow from the following fundamental theorem.

Fundamental Theorem of Algebra/Definition: Every degree n polynomial factors as

$$a(x - c_1)^{k_1}(x - c_2)^{k_2} \cdots (x - c_m)^{k_m}$$

where $c_1, \ldots, c_m \in \mathbb{C}$ are distinct, $k_1 + k_2 + \cdots + k_m = n$, and $0 \neq a \in \mathbb{C}$. The c_i are called the **roots** of the polynomial and k_i is called the **algebraic multiplicity** of the root c_i.

Fact: If $A \in M_{n\times n}(\mathbb{F})$ then the characteristic polynomial of A is a polynomial of degree n.

So...when counted correctly, $A \in M_{n\times n}(\mathbb{F})$ has exactly n eigenvalues.

Example: The matrix A from the previous example has size 3×3 and has 3 eigenvalues (counted correctly):

Diagonalization

Recall that we want diagonal matrices for our matrix representations!

An Old Example: Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{2(x+y+z)}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We use the two bases for $\mathbb{R}^3$:

- $\mathcal{B}$ is the standard basis
- $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Then

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} \quad \text{and} \quad [T]_{\mathcal{C}}^{\mathcal{C}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover,

Definition: A square matrix A is said to be **diagonalizable** if there exists an invertible matrix P such that

Definition: Two matrices A and B in $M_{n\times n}(\mathbb{F})$ are said to be **similar** if

Theorem: If A and B are similar matrices, then they have the same determinant, same eigenvalues, same rank, and same trace.