Direct Sums and Linear Independence

Recall: Last class we defined a special subspace called the direct sum. We now look at this a bit more carefully.

Theorem: Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of the vector space V over the field $\mathbb{F}$. Then $V = \mathcal{U} \oplus \mathcal{W}$ if and only if $V = \mathcal{U} + \mathcal{W}$ and

$$\mathcal{U} \cap \mathcal{W} = \{\mathbf{x} \in V : \mathbf{x} \in \mathcal{U} \text{ and } \mathbf{x} \in \mathcal{W}\} = \{\mathbf{0}\}.$$

Note: Consequently, from our definition of direct sum last class, if $V = \mathcal{U} \oplus \mathcal{W}$ then *every* element $\mathbf{x} \in V$ can be written *uniquely* in the form $\mathbf{x} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in \mathcal{U}$ and $\mathbf{w} \in \mathcal{W}$.

Example: Let $V = \mathbb{R}^3$,

$$\mathcal{U} = Span\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}\right) \subseteq \mathbb{R}^3 \quad \text{and} \quad \mathcal{W} = Span\left(\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}\right) \subseteq \mathbb{R}^3.$$

Linear Independence

Motivating Example: Working over $\mathbb{F} = \mathbb{R}$, note that

$$\left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} = Span(\{\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_4}\})$$

Definition: A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ in a vector space V over the field $\mathbb{F}$ is **linearly independent** if the only solution to the equation

$$a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k} = \mathbf{0}$$

Examples:

1. The set

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$$

   is

2. $\{1+x, 1\}$ are linearly independent in $\mathcal{P}_1(\mathbb{R})$ since

3. Is $\{x+x^2-2x^3, 2x-x^2+x^3, x+5x^2+3x^3\}$ linearly independent in $\mathcal{P}_3(\mathbb{R})$?

Proposition: If $\mathbf{0} \in \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$, then $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is a linearly dependent set of vectors.

Corollary: If $\mathcal{U}$ is a subspace of a vector space V then $\mathcal{U}$ is a linearly dependent set of vectors.

Proposition: Let $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ be a set of distinct vectors. $\mathcal{B}$ is linearly dependent if and only if there is a vector $\mathbf{v}_j$ in $\mathcal{B}$ such that $\mathbf{v_j}$ is a linear combination of $\{\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}, \mathbf{v_{j+1}}, \ldots, \mathbf{v_k}\}$ (i.e., $\mathbf{v_j} \in Span(\{\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}, \mathbf{v_{j+1}}, \ldots, \mathbf{v_k}\})$).

Theorem: Let V be a vector space over $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\} \subseteq V$ and set $\mathcal{U} = Span(\mathcal{B})$. Then there exists a subset $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}'$ is linearly independent and $\mathcal{U} = Span(\mathcal{B}) = Span(\mathcal{B}')$.