# Diagonalization

**Question:** When is a matrix $A \in M_{n\times n}(\mathbb{F})$ diagonalizable? Recall that we need a basis of eigenvectors!

**Theorem:** A matrix $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable if and only if there exists a basis $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ of $\mathbb{F}^n$ such that each $\mathbf{v_j}$ is an eigenvector of $A$. Moreover, if such a basis exists then $P^{-1}AP = D$ with

$$P = \begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where $\mathbf{v_j}$ is an eigenvector with eigenvalue $\lambda_j$.

**Examples:**

1. The matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. A basis for the eigenspace $E_{\lambda_1}$ is

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and a basis for the eigenspace $E_{\lambda_2}$ is

$$\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.$$

2. The matrix

$$A = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

is diagonalizable! We have,

3. The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is diagonalizable! We have,

**Theorem:** Let $\lambda_1, \ldots, \lambda_m$ be *distinct* eigenvalues of $A \in M_{n\times n}(\mathbb{F})$ with corresponding eigenvectors $\mathbf{v_1}, \ldots, \mathbf{v_m}$. Then $\{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$ is linearly independent.

**Theorem:** If $A \in M_{n\times n}(\mathbb{F})$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

**Example:** A matrix $A \in M_{4\times 4}(\mathbb{C})$ with characteristic polynomial

$$(\lambda - i)(\lambda + i)(\lambda - 1)(\lambda - (3 + i))$$

**Theorem:** Let $\lambda_1, \ldots, \lambda_k$ be *distinct* eigenvalues of $A \in M_{n\times n}(\mathbb{F})$ and for each $j$ let the eigenspace $E_{\lambda_j}$ have basis $\{\mathbf{v_{j,1}}, \ldots, \mathbf{v_{j,m_j}}\}$ (so that $\dim(E_{\lambda_j}) = m_j$). Then

$$\{\mathbf{v_{1,1}}, \ldots, \mathbf{v_{1,m_1}}, \mathbf{v_{2,1}}, \ldots, \mathbf{v_{2,m_2}}, \ldots, \mathbf{v_{k,1}}, \ldots, \mathbf{v_{k,m_k}}\}$$

is linearly independent in $\mathbb{F}^n$.