Diagonalization: Conclusion And Applications

Note: Let $A \in M_{n\times n}(\mathbb{F})$. Find bases for each eigenspace of A. Last class we saw that the collection of all these bases is linearly independent. So the entire collection is a basis consisting of eigenvectors for $\mathbb{F}^n$ if and only if the sum of the dimensions of the eigenspaces is n. That is, A is diagonalizable if and only if the sum of the dimensions of its eigenspaces is n.

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$. The characteristic polynomial of A is

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2.$$

Definition: Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$.

1. The **algebraic multiplicity** of λ is the algebraic multiplicity of λ as a root of the characteristic polynomial of A.

2. The **geometric multiplicity** of λ is

Proposition: Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. Then

Theorem: $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable if and only if every eigenvalue of A has its geometric multiplicity equal to its algebraic multiplicity.

Examples:

1. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$

2. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$

3. $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix} \in M_{4\times 4}(\mathbb{C})$

4. $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$

Applications Of Diagonalization

I - Powers Of Matrices

Recall:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_k \end{bmatrix} \implies D^n = \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_k^n \end{bmatrix}.$$

Example: The matrix $A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$ is diagonalizable. We have $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

II - Decoupling Differential Equations

Example: Consider the following system of differential equations

$$\frac{dx}{dt} = 5x - 3y$$
$$\frac{dy}{dt} = -6x + 2y$$

For the sake of notation, let

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}.$$

Then

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Note that A is diagonalizable. We have $P^{-1}AP = D$ where

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}.$$