Coordinates and Linear Transformations

Recall: Last class we saw that $M_{2\times 2}(\mathbb{R})$ and $\mathbb{R}^4$ "look the same" via coordinate vectors.

Example: $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$ also "look the same". To see this, recall that

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$ and

$$\mathcal{C} = \{1, x, x^2\}$$

is a basis for $\mathcal{P}_2(\mathbb{R})$.

Coordinate vectors behave nicely as follows:

Theorem: Let V be a vector space over the field $\mathbb{F}$ with basis $\mathcal{B}$. Then for all $\mathbf{x}, \mathbf{y} \in V$ and scalars $t \in \mathbb{F}$, we have

Remark: If V is n-dimensional then we can define the function $T : V \to \mathbb{F}^n$ by the rule

$$T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}.$$

The previous theorem says that

Linear Transformations

Goal: We want to start comparing vector spaces (such as $\mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}^3$). To do this, we use special functions that behave nicely with our vector addition and scalar multiplication operations!

Definition: Let V and W be vector spaces over the field $\mathbb{F}$. A **linear transformation** T from V to W, denoted $T : V \to W$, is

Examples:

1. Define $tr : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ by the function

2. Define $D : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ by differentiation:

$$D(a + bx + cx^2 + dx^3) = (a + bx + cx^2 + dx^3)' = b + 2cx + 3dx^2.$$

3. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xyz \\ 0 \\ 0 \end{bmatrix}$$

Properties of Linear Transformations: We now look at some of the basic properties of linear transformations.

Proposition: Let $T : V \to W$ be a linear transformation. Then

$$T(\mathbf{0_V}) = \mathbf{0_W}.$$

Proposition: Let $T : V \to W$ be a linear transformation. For any vectors $\mathbf{v_1}, \dots, \mathbf{v_n} \in V$ and scalars $a_1, \dots, a_n \in \mathbb{F}$ we have

$$T(a_1\mathbf{v_1} + \cdots + a_n\mathbf{a_n}) = a_1T(\mathbf{v_1}) + \cdots + a_nT(\mathbf{v_n}).$$

The next fact needs some preliminary notation.

Notation: Let $T : V \to W$ be a linear transformation and $E \subseteq V$. Then

$$T(E) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \quad \text{for some } \mathbf{v} \in E\}.$$

Proposition: Let $T : V \to W$ be a linear transformation and $E \subseteq V$. Then

$$T(Span(E)) = Span(T(E)).$$

Proposition: If $T : V \to W$ and $S : W \to U$ are linear transformations, then the composite function

$$S \circ T : V \xrightarrow{T} W \xrightarrow{S} U$$

is a linear transformation.