Bases and Dimension

Motivating Examples: Let's look at three familiar examples:

1. $\mathbb{R}^2 = Span\left(\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\right)$
2. $P_n(\mathbb{R}) = Span(\{1, x, x^2, \ldots, x^n\})$
3. $M_{2\times 2}(\mathbb{R}) =$

Definition: A vector space V is said to be **finite-dimensional** if there exists a *finite* set of vectors $\mathcal{B}$ such that

Examples: $\mathbb{R}^k, \mathbb{C}^k, P_n(\mathbb{R})$ and $M_{m\times n}(\mathbb{R})$ are all examples of finite-dimensional vector spaces.

Theorem: Let V be a finite-dimensional vector space. Then there exists a finite set $\mathcal{B}$ of vectors such that

1. $V = Span(\mathcal{B})$;
2. $\mathcal{B}$ is linearly independent.

Definition: A **basis** for a vector space V is a set $\mathcal{B}$ of vectors in V such that

1. $V = Span(\mathcal{B})$;
2. $\mathcal{B}$ is

Examples:

1.

2. $\{1, x, x^2, \ldots, x^n\}$ is the *standard basis* for $P_n(\mathbb{R})$

3.

4. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$

5. $\{1, x-1, (x-2)(x-1)\}$ is a basis for $P_2(\mathbb{R})$:

Proposition: Let V be a vector space such that $V = Span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$. If $\mathcal{B} = \{\mathbf{u_1}, \ldots, \mathbf{u_k}\}$ is a linearly independent set of vectors in V, then $k \leq n$.

Theorem: Suppose $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ and $\mathcal{C} = \{\mathbf{u_1}, \ldots, \mathbf{u_k}\}$ are both bases of a vector space V. Then $k = n$.

Definition: The **dimension** of a vector space over the field $\mathbb{F}$, denoted $\dim_{\mathbb{F}}(V)$ or $\dim(V)$, is the number of vectors in any basis for V.

Note: If there is no finite basis for V, then we say that V is

Examples:

1. $\dim(\mathbb{R}^n) =$
2. $\dim(\mathcal{P}_n(\mathbb{R})) =$

3. $\dim(M_{m\times n}(\mathbb{R})) =$

4. $\mathcal{P}(\mathbb{R})$

5. The set
$$\mathcal{U} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a+b+c+d=0 \right\}$$
is a subspace of $M_{2\times 2}(\mathbb{R})$