Bases and Dimension and Coordinates (Continued)

Recall: Let $V = Span(\{\mathbf{v_1}, \ldots, \mathbf{v_k}\})$ be a vector space. If one vector, say $\mathbf{v_1}$, is a linear combination of the other vectors, then we can remove $\mathbf{v_1}$ and $V = Span(\{\mathbf{v_2}, \ldots, \mathbf{v_k}\})$. If $\mathbf{v_2}$ is a linear combination of $\mathbf{v_3}, \ldots, \mathbf{v_k}$ then we can repeat and remove $\mathbf{v_2}$ so that $V = Span(\{\mathbf{v_3}, \ldots, \mathbf{v_k}\})$. Continuing in this way, we have that the process must stop (since we have only a finite number of vectors in our spanning set) and when it does the remaining vectors are linearly independent and span V. That is, when the process stops we have a basis for V. Now instead of reducing our set of vectors, let us build it up!

Basis Extension Theorem: Let V be a finite-dimensional vector space and $\mathbf{v_1}, \ldots, \mathbf{v_m}$ be linearly independent vectors in V. Then there exist $n = \dim V - m$ vectors $\mathbf{u_1}, \ldots, \mathbf{u_n}$ in V such that $\{\mathbf{v_1}, \ldots, \mathbf{v_m}, \mathbf{u_1}, \ldots, \mathbf{u_n}\}$ is a basis for V.

Theorem: Let V be a finite-dimensional vector space and $\mathcal{U}$ be a subspace. Then $\dim\ \mathcal{U} \leq \dim V$.

The Basis Extension Theorem makes it easier to verify one has a basis for a finite-dimensional vector space via the following fact:

Theorem: Let V be a finite-dimensional vector space of dimension n.

1. If $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\} \subseteq V$ is a linearly independent set, then $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis of V.
2. If $Span(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}) = V$, then $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis of V.

Corollary: Let V be an n-dimensional vector space and $\mathcal{U}$ be a subspace of V. If dim $\mathcal{U} = n$, then $\mathcal{U} = V$.

Exercise: Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be subspaces of a finite-dimensional vector space V. Then

$$\dim(\mathcal{U}_1 + \mathcal{U}_2) = \dim(\mathcal{U}_1) + \dim(\mathcal{U}_2) - \dim(\mathcal{U}_1 \cap \mathcal{U}_2).$$

Question: A vector space can have many different bases. Which basis should we use?

Theorem: Let V be a vector space with basis $\{\mathbf{v_1}, \dots, \mathbf{v_n}\}$. Then *any* vector $\mathbf{x} \in V$ can be written *uniquely* as a linear combination

$$\mathbf{x} = a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n}$$

with scalars $a_1, \dots, a_n$.

Definition: Let $\mathcal{B} = \{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ be an (ordered) basis for a vector space V. If $\mathbf{x}$ is a vector in V written as

$$\mathbf{x} = a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n},$$

then the **coordinate vector of x with respect to** $\mathcal{B}$ is

Examples:

1. The sets

$$\mathcal{B} = \{1, x, x^2\}, \quad \mathcal{C} = \{1, x^2, x\}, \quad \mathcal{D} = \{1, 1+x, 1+x+x^2\}$$

are all bases of $\mathcal{P}_2(\mathbb{R})$.

2. The set

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \right\}$$

is a basis for $M_{2\times 2}(\mathbb{R})$. Let

$$\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

What is $[\mathbf{x}]_{\mathcal{B}}$?