MATH 721, Algebra II
Exercises 7
Due Wed 04 Mar

Throughout this homework set, let R be a non-zero commutative ring with identity, and let A, B, C be matrices with entries in R such that A and B are square.

Exercise 1. Assume that R is a field. Prove that the following conditions are equivalent:

(i) A is invertible.
(ii) The columns of A are linearly independent.
(iii) The rows of A are linearly independent.

Exercise 2. Assume that A, B, and C fit into a block matrix $T = \left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right)$. Prove that $\det(T) = \det(A)\det(B)$. (Hint: Expand along the top row and use induction.)

Exercise 3. Let $\mathrm{GL}_n(R)$ denote the set of invertible $n \times n$ matrices with entries in R; this is a "general linear" group. Let $\mathrm{SL}_n(R)$ denote the set of $n \times n$ matrices with entries in R that have determinant $= 1$; this is a "special linear" group. Let $R^\times$ denote the set of units of R, which is an abelian group under multiplication.

(a) Prove that $\mathrm{GL}_n(R)$ is a group under matrix multiplication.
(b) Prove that $\mathrm{SL}_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$ and $\mathrm{GL}_n(R)/\mathrm{SL}_n(R) \cong R^\times$.