Problem Set 4

Due: 9:00 a.m. on Thursday, December 15

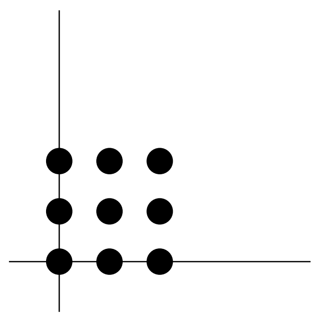
Instructions: The following set of questions are based on the Course Project Presentations. Work *one* of the following problems which is not based on your own project. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets.* Submit your solution by email to Dr. Cooper (as an attached pdf file).

Exercises:

1. Suppose I and J are Borel-fixed ideals in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. Prove that $I \cap J$ and $I + J$ are Borel-fixed.

2. Let $S = k[x_1, x_2]$ and $R = k[x_0, x_1, x_2]$ with the standard grading.

   (a) Does the given figure represent a complete intersection, rectangular complete intersection, or neither? If it is a complete intersection, determine the type. If it is a rectangular complete intersection, determine the type and list the bijections.

   (b) Recall that C.I.(d_1, d_2) denotes the set of all finite sets of distinct points in $\mathbb{P}^2$ which are a complete intersection of type $\{d_1, d_2\}$. A key step in proving Proposition 4.2 of Megan's paper is the *Cayley-Bacharach Theorem* which relates the Hilbert functions of two subsets of complete intersections as follows: If $\mathbb{X} \in$ C.I.(d_1, d_2) and $\mathbb{Y} \subseteq \mathbb{X}$, then

   $$\Delta H(\mathbb{X}, t) = \Delta H(\mathbb{Y}, t) + \Delta H(\mathbb{X} \setminus \mathbb{Y}, d_1 + d_2 - 2 - t).$$

   (This generalizes in $\mathbb{P}^n$).

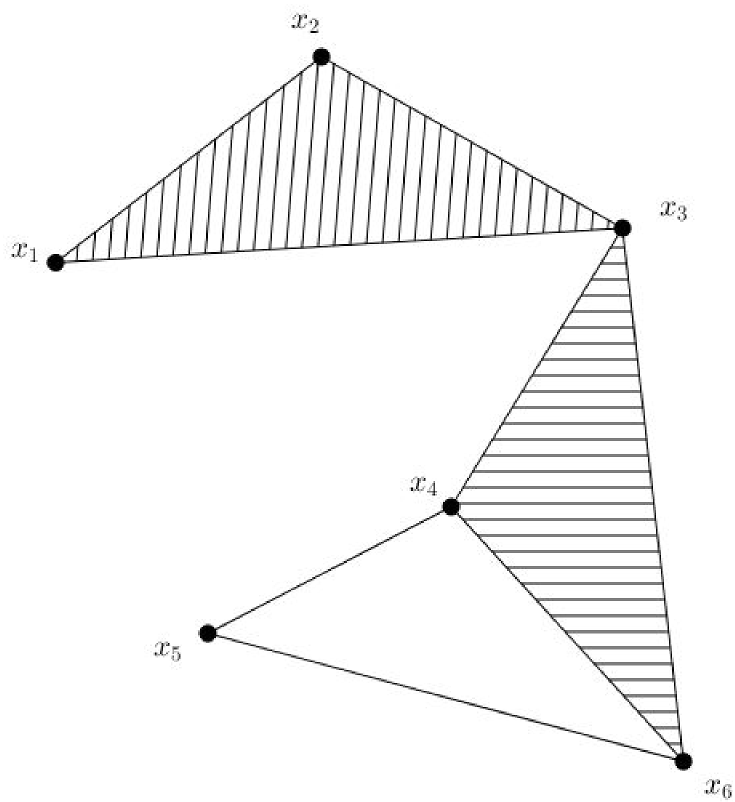
   Use the Cayley-Bacharach Theorem to prove the *Classical Cayley-Bacharach Theorem*: If $\mathbb{X} = \{P_1, \ldots, P_9\}$ is the complete intersection of two cubics in $\mathbb{P}^2$ (i.e., $\mathbb{X} \in$ C.I.$(3, 3)$), then any cubic passing through 8 of the 9 points of $\mathbb{X}$ must also pass through the remaining 9th point.

3. Let $R = k[x_1, \ldots, x_n]$ and let $\mathbb{A} = \{a_1, \ldots, a_n\}$ be a sequence of degrees. Then an ideal L is said to be $\mathbb{A}$ lex-plus-powers (or lex-plus-powers with respect to $\mathbb{A}$) if L is minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}$ and monomials $m_1, \ldots m_p$ such that whenever $r \in R_{\deg(m_i)}$ and $r \geq_{\text{lex}} m_i$, then $r \in L$. We denote by $\mathcal{L}_{\mathcal{H},\mathbb{A}}$ the unique $\mathbb{A}$ lex-plus-powers ideal with Hilbert function $\mathcal{H}$ (if such an ideal exists).

Suppose that $\mathbb{A} \leq \mathbb{B}$ for degree sequences $\mathbb{A} = \{a_1, \ldots, a_n\}$ and $\mathbb{B} = \{b_1, \ldots, b_n\}$, that $\mathcal{H}$ is a Hilbert function, and that $\mathcal{L}_{\mathcal{H},\mathbb{A}}$ exists. If $\mathbb{L}$ is the sequence of degrees of the regular sequence of pure powers in the lex ideal attaining $\mathcal{H}$ and $\mathbb{B} \leq \mathbb{L}$, then $\mathcal{L}_{\mathcal{H},\mathbb{B}}$ exists.

Given $\mathbb{A} = \{3,3\}$, $\mathbb{B} = \{3,4\}$, and $\mathbb{L} = \{3,5\}$, where $\mathcal{H} = (1,2,3,2,1)$, and if we know $\mathcal{L}_{\mathcal{H},\mathbb{A}}$ exists, find the ideal $\mathcal{L}_{\mathcal{H},\mathbb{B}}$.

4. Let $\Delta =$



Mentally check that Δ is a simplicial complex, then find/compute the following:

(a) $\dim(\Delta)$,

(b) I_Δ,

(c) $f(\Delta)$,

(d) $H(k[\Delta], d)$ for $1 \leq d \leq 4$, using the proposition from Dylan's presentation (Proposition 12 in the paper), and

(e) $h(\Delta)$.

Is Δ Cohen-Macaulay?

5. Using the notation set in Corey's project, let $M \subseteq \mathcal{M}_{\underline{e}}$. Prove that M is an order ideal (of monomials) if and only if M is closed.