Problem Set 9
Due: 3:00 p.m. on Wednesday, October 28

Instructions: Carefully read Sections 10.3 and 10.4 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets.* Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, unless otherwise stated assume that R is a ring with 1 and that M is a left R-module.

1. (Dummit and Foote §10.3 #7) Let N be a submodule of M. Prove that if both M/N and N are finitely generated then so is M.
2. (Dummit and Foote §10.3 #13) Let R be a commutative ring and let F be a free R-module of finite rank. Prove the following isomorphism of R-modules: $\text{Hom}_R(F, R) \cong F$.
3. A non-zero unitary R-module M is called *simple* if its only submodules are 0 and M.
   - (i) Prove that if M is a simple R-module, then M is cyclic.
   - (ii) Let $\alpha : M \to N$ be a homomorphism between simple R-modules. Prove that α is either 0 or an isomorphism.
   - (iii) Assume that R is commutative. Prove that M is a simple R-module if and only if there is a maximal ideal $I \subseteq R$ such that $M \cong R/I$ (as R-modules).
4. (Dummit and Foote §10.4 #16) Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.
   - (a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes (r \mod J)$.
   - (b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \mod I) \otimes (r' \mod J)$ to $rr' \mod (I + J)$.
5. Let R be a ring with identity and let M be a unital right R-module.
   - (i) Prove that every element of $M \otimes_R R$ can be written as a simple tensor of the form $m \otimes_R 1$.
   - (ii) Prove that there is an Abelian group isomorphism $F : M \otimes_R R \to M$ such that $F(m \otimes_R r) = mr$ for all $m \in M$ and $r \in R$.