Problem Set 8

Due: 3:00 p.m. on Wednesday, October 21

Instructions: Carefully read Sections 10.1 and 10.2 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that R is a ring with 1 and that M is a left R-module.

1. (Dummit and Foote §10.1 #5) For any left ideal I of R define

$$IM = \left\{ \sum_{\text{finte}} a_i m_i \mid a_i \in I, m_i \in M \right\}$$

   to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.

2. (Dummit and Foote §10.1 #8) An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

   (a) Prove that if R is an integral domain then $\text{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).

   (b) Give an example of a ring R and an R-module M such that $\text{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]

   (c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

3. (Dummit and Foote §10.2 #1) Use the submodule criterion to show that kernels and images of R-module homomorphisms are submodules.

4. (Dummit and Foote §10.2 #9) Let R be a commutative ring. Prove that $\text{Hom}_R(R, M)$ and M are isomorphic as left R-modules. [Show that each element of $\text{Hom}_R(R, M)$ is determined by its value on the identity of R.]

5. (Dummit and Foote §10.2 #13) Let I be a nilpotent ideal in a commutative ring R (cf. Exercise 37, Section 7.3), let M and N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Show that if the induced map $\overline{\varphi} : M/IM \to N/IN$ is surjective, then φ is surjective.