Problem Set 5

Due: 3:00 p.m. on Wednesday, September 30

Instructions: Carefully read Sections 8.1 and 8.2 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, we denote a greatest common divisor of elements a and b by $\text{g.c.d.}(a, b)$.

1. (Dummit and Foote §8.1 #3) Let R be a Euclidean Domain. Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of R of norm m is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.
2. (Dummit and Foote §8.1 #4 (a)) Let R be a Euclidean Domain. Prove that if $\text{g.c.d.}(a, b) = 1$ and a divides bc, then a divides c. More generally, show that if a divides bc with nonzero a, b then $\dfrac{a}{\text{g.c.d.}(a, b)}$ divides c.
3. Find a generator for the ideal $(85, 1 + 13i)$ in $\mathbb{Z}[i]$.
4. In class we proved that $I = (2, 1 + \sqrt{-5})$ is not a principal ideal of $\mathbb{Z}[\sqrt{-5}]$. Prove that $J = (3, 2 - \sqrt{-5})$ is also not principal yet IJ is a principal ideal in $\mathbb{Z}[\sqrt{-5}]$.
5. (Dummit and Foote §8.2 #1) Prove that in a Principal Ideal Domain two ideals (a) and (b) are comaximal if and only if a greatest common divisor of a and b is 1.
6. (Dummit and Foote §8.2 #3) Prove that a quotient of a P.I.D. by a prime ideal is again a P.I.D.
7. (Dummit and Foote §8.2 #8) Prove that if R is a Principal Ideal Domain and D is a multiplicatively closed subset of R, then $D^{-1}R$ is also a P.I.D.