Problem Set 4

Due: 3:00 p.m. on Wednesday, September 23

Instructions: Carefully read Sections 7.4, 7.5, and 7.6 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that all rings are non-zero and contain an identity $1 \neq 0$.

1. (Dummit and Foote §7.4 #7) Let R be a commutative ring. Prove that the principal ideal generated by x in the polynomial ring $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.

2. (Dummit and Foote §7.4 #11) Assume R is a commutative ring. Let I and J be ideals of R and assume P is a prime ideal of R that contains IJ. Prove that either I or J is contained in P.

3. (Dummit and Foote §7.4 #26) Let R be a commutative ring. Recall that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{Z}^+$. For this exercise you may assume the fact that the set of nilpotent elements form an ideal – called the *nilradical* of R. Prove that a prime ideal in R contains every nilpotent element. Deduce that the nilradical of R is contained in the intersection of all the prime ideals of R.

4. A commutative ring R is called a *local ring* if it has a unique maximal ideal. Prove that if R is a local ring with maximal ideal M then every element of $R - M$ is a unit.

5. Let R be a commutative ring and let S be a multiplicatively closed subset of R. Let I and J be ideals of R. Prove the following equalities:

   (a) $S^{-1}(I + J) = S^{-1}I + S^{-1}J$;

   (b) $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$.

6. (Dummit and Foote §7.6 #3) Let R and S be rings. Prove that every ideal of $R \times S$ is of the form $I \times J$ where I is an ideal of R and J is an ideal of S.