Problem Set 2

Due: 3:00 p.m. on Wednesday, September 9

Instructions: Carefully read Section 7.3 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that all rings are non-zero and contain an identity $1 \neq 0$.

1. (Dummit and Foote #7) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{Z} \right\}$ be the subring of $M_2(\mathbb{Z})$ of upper triangular matrices. Prove that the map $\varphi : R \to \mathbb{Z} \times \mathbb{Z}$ defined by $\varphi : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (a, d)$ is a surjective homomorphism and describe its kernel.

2. (Dummit and Foot #10) Decide which of the following are ideals of the ring $\mathbb{Z}[x]$:
   (i) the set S of all polynomials whose constant term is a multiple of 3
   (ii) $S = \mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of x appear)
   (iii) the set S of polynomials whose coefficients sum to zero
   (iv) the set S of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

3. (Dummit and Foote #13) Prove that the ring $M_2(\mathbb{R})$ contains a subring that is isomorphic to $\mathbb{C}$.

4. (Dummit and Foote #16) Let $\varphi : R \to S$ be a surjective homomorphism of rings. Prove that the image of the center of R is contained in the center of S.

5. (Dummit and Foote #17) Let R and S be non-zero rings with identity and denote their respective identities by 1_R and 1_S. Let $\varphi : R \to S$ be a non-zero homomorphism of rings.
   (a) Prove that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is a zero divisor in S. Deduce that if S is an integral domain then every ring homomorphism from R to S sends the identity of R to the identity of S.
   (b) Prove that if $\varphi(1_R) = 1_S$ then $\varphi(u)$ is a unit in S and that $\varphi(u^{-1}) = \varphi(u)^{-1}$ for each unit u of R.

6. Show that for $D = 6$ that the group of units $\mathcal{O}^{\times}$ of the quadratic integer ring $\mathcal{O}$ is infinite by exhibiting an explicit unit of infinite (multiplicative) order in the ring.

7. Let $f : R \to S$ be a homomorphism of rings. Suppose that $I \subset R$ is an ideal. Define $\overline{f} : R/I \to S$ by $\overline{f}(r + I) = f(r)$. Prove that $\overline{f}$ is a well-defined ring homomorphism if and only if $I \subseteq \operatorname{Ker}(f)$.