Problem Set 10
Due: Wednesday, November 7

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined in the *General Problem Set Guidelines Sheet*.

Unless otherwise stated, all problems can be found in the appropriate *Exercises* sections of the text (*Abstract Algebra* by D. Dummit and R. Foote, 3rd Edition).

- Section 13.4 # 2, 3, 4, 5
- Using the following steps, determine the degree of $\beta := 1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.
  (a) Let $\alpha = \sqrt[3]{2}$. Show that $\beta \in \mathbb{Q}(\alpha)$.
  (b) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
  (c) Explain why $[\mathbb{Q}(\beta) : \mathbb{Q}]$ divides $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
  (d) Show that $[\mathbb{Q}(\beta) : \mathbb{Q}] \neq 1$.
  (e) What is the degree of β over $\mathbb{Q}$?
- Let $\zeta \neq 1$ be any nontrivial ninth root of unity such that $\omega = \zeta + \zeta^{-1} \neq -1$.
  (a) Using a cyclotomic polynomial, show that $\zeta^8 + \zeta^7 + \cdots + \zeta + 1 = 0$.
  (b) Show that $0 = \zeta^8 + \zeta^7 + \cdots + \zeta + 1 = \omega^4 + \omega^3 - 3\omega^2 - 2\omega + 1$.
  (c) Observe that $\omega^4 + \omega^3 - 3\omega^2 - 2\omega + 1 = (\omega + 1)(\omega^3 - 3\omega + 1)$. Why is ω a root of the polynomial $x^3 - 3x + 1$?
  (d) Find three distinct roots of $x^3 - 3x + 1$.
  (e) Prove that the splitting field of $x^3 - 3x + 1$ over $\mathbb{Q}$ is $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$ where $\zeta_9 = e^{\frac{2\pi i}{9}}$.