Problem Set 6

Due: At the Beginning of Class on Thursday, April 24

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem and Quiz Sets*.

(1) (Gallian, Chapter 9 Exercises, #34) In $\mathbb{Z}$, let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $\mathbb{Z} = HK$. Does $\mathbb{Z} = H \times K$? Why or why not?

(2) (Gallian, Chapter 9 Exercises, #57) Give an example of subgroups H and K of a group G such that HK is not a subgroup of G.

(3) (Gallian, Chapter 10 Exercises, #8) Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this homomorphism allow you to conclude that A_n is a normal subgroup of S_n of index 2?

(4) (Gallian, Chapter 10 Exercises, #12) Suppose that k is a divisor of n. Prove that $\mathbb{Z}_n/\langle k \rangle \approx \mathbb{Z}_k$.

(5) (Gallian, Chapter 10 Exercises, #13) Prove that $(A \oplus B)/(A \oplus \{e\}) \approx B$.

(6) (Gallian, Chapter 10 Exercises, #15) Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and Ker $\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9. Fully justify your answer.

(7) (Gallian, Chapter 10 Exercises, #29) Suppose that there is a homomorphism from a finite group G onto $\mathbb{Z}_{10}$. Prove that G has normal subgroups of indexes 2 and 5.