Tutorial Worksheet #9
Friday, March 20

Instructions: Work through the following exercises. *Fully show and explain all of your work.* Remember to use good notation and full sentences.

Exercises: From the textbook *Linear Algebra And Its Applications*, fifth edition, by David C. Lay, Steven R. Lay, Judi J. McDonald.

1. Fun with Complex Numbers! Let $z = \sqrt{3} + i$ and $w = 1 - 2i$. Find:
   (a) $z + w$
   (b) zw
   (c) $\overline{z}$
   (d) $z\overline{z}$
   (e) $|z|$
   (f) $\dfrac{w}{z}$
   (g) z^4 (use De Moivre's Theorem)
2. (§5.1 #4) Is $\mathbf{x} = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$? If so, find the eigenvalue.
3. (§5.1 #7) Is $\lambda = 4$ an eigenvalue of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.
4. (§5.1 #11) Find a basis for the eigenspace of $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 10$.
5. (§5.1 #15) Find a basis for the eigenspace of $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$.
6. (§5.1 #19) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.
7. (§5.1 #24) Construct an example of a 2×2 matrix with only one distinct eigenvalue.
8. (§5.1 #25) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint:* Suppose a non-zero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]
9. (§5.1 #26) Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

10. (§5.1 #27) Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint:* Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

11. (§5.1 #35) Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of a 2×2 matrix A with corresponding eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$. Let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Find $T(\mathbf{w})$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

12. (§5.2 #3) Find the characteristic polynomial and the eigenvalues of $A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$.

13. (§5.2 #7) Find the characteristic polynomial and the eigenvalues of $A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$.

14. (§5.2 #9) Find the characteristic polynomial (using a cofactor expansion) of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$.

15. (§5.2 #15) List the eigenvalues, repeated according to their multiplicities, for

$$A = \begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

16. (§5.2 #19) Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicities, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

Explain why $\det A$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

17. (§5.2 #20) Use a property of determinants to show that A and A^T have the same characteristic polynomial.

18. (§5.2 #24) Show that if A and B are similar matrices, then $\det A = \det B$.