Tutorial Worksheet #4
Friday, February 7

Instructions: Work through the following exercises. *Fully show and explain all of your work.* Remember to use good notation and full sentences.

Exercises: From the textbook *Linear Algebra And Its Applications*, fifth edition, by David C. Lay, Steven R. Lay, Judi J. McDonald.

1. (§1.8 #3) Define T by $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$. Find a vector $\mathbf{x}$ whose image

   under T is $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$, and determine whether $\mathbf{x}$ is unique.

2. (§1.8 #9) Find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the matrix
$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}.$$

3. (§1.8 #11) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 2. Is $\mathbf{b}$ in the range of the

   linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

4. (§1.8 #17) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

   maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $3\mathbf{u}, 2\mathbf{v}$ and $3\mathbf{u} + 2\mathbf{v}$.

5. (§1.8 #19) Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y_1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\mathbf{y_2} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be

   a linear transformation that maps $\mathbf{e_1}$ into $\mathbf{y_1}$ and $\mathbf{e_2}$ into $\mathbf{y_2}$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and

   $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

6. (§1.8 #24) Suppose vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose that $T(\mathbf{v_i}) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

7. (§1.8 #29) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

   (a) Show that f is a linear transformation when $b = 0$.

   (b) Find a property of a linear transformation that is violated when $b \neq 0$.

   (c) Why is f called a linear transformation?

8. (§1.8 #32) Show that the transformation T defined by

$$T\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = \left[\begin{array}{c} 4x_1 - 2x_2 \\ 3|x_2| \end{array}\right]$$

is not linear.

9. (§1.8 #33) Show that the transformation T defined by

$$T\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = \left[\begin{array}{c} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{array}\right]$$

is not linear.

10. (§1.8 #35) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation.

11. (§1.9 #3) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points (about the origin) through $3\pi/2$ radians (counter-clockwise).

12. (§1.9 #9) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first performs a horizontal shear that transforms $\mathbf{e_2}$ into $\mathbf{e_2} - 2\mathbf{e_1}$ (leaving $\mathbf{e_1}$ unchanged) and then reflects points through the line $x_2 = -x_1$.

13. (§1.9 # 15) Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables:

$$\left[\begin{array}{ccc} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{c} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{array}\right].$$

14. (§1.9 #27) Determine if the linear transformation defined by

$$T(x_1, x_2 x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$

is one-to-one and/or onto. Justify each answer.