Cross Product, Lines and Planes

§I – Cross Product

Definition: Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be in $\mathbb{R}^3$. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is

Example: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Theorem: If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in $\mathbb{R}^3$, then:

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
2.
3. $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$
4.
5. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$

Theorem: Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in $\mathbb{R}^3$ and k be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2.
3. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} + \mathbf{w})$
4.
5. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
6.

Warning: In general,

Theorem: If $\mathbf{u}, \mathbf{v}$ are in $\mathbb{R}^3$, then

II - Lines and Planes:

Recall:

1. Let L be a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ that contains the point x_0 and is parallel to the vector $\mathbf{v} \neq \mathbf{0}$. Then an equation of the line through x_0 and parallel to $\mathbf{v}$ is

2. Let W be a plane in $\mathbb{R}^3$ that contains the point x_0 and is parallel to the noncollinear vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. Then an equation of the plane through x_0 that is parallel to $\mathbf{v_1}$ and $\mathbf{v_2}$ is

Examples:

1. Find vector and parametric equations of the line in $\mathbb{R}^3$ passing through $P_0(1, 2, -3)$ and parallel to $\mathbf{v} = (4, -5, 1)$.

2. Find vector and parametric equations of the plane $x - y + 2z = 5$.

Theorem: The vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ lie in the same plane in $\mathbb{R}^3$ if and only if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Example: Find an equation of the plane in $\mathbb{R}^3$ containing the points $P(1, 0, 2), Q(-1, 1, 2)$ and $R(5, 0, 3)$.