Chapter 6: Orthogonality and Least Squares

§6.1 – Inner Product, Length, and Orthogonality

Goal: To study the Euclidean Vector Spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ more deeply.

Basic Terminology:

Definitions: Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be in $\mathbb{R}^n$.

1. The **inner product** (or **dot product**) of $\mathbf{u}$ and $\mathbf{v}$ is

2. The **length** (or **norm**) of $\mathbf{u}$ is

3. The **distance** between $\mathbf{u}$ and $\mathbf{v}$ is

Examples: Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix}$.

Theorem: Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be in $\mathbb{R}^n$ and c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v}$

2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$

3. $(c\mathbf{u}) \cdot \mathbf{v}$

4. $\mathbf{u} \cdot \mathbf{u}$

5. $||\mathbf{u}||$

6. $||\mathbf{u}|| = 0$

7. $||c\mathbf{u}||$

Example: Let $\mathbf{u}, \mathbf{v}$ be in $\mathbb{R}^n$. Then

$$(\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v})$$

Note: If $\mathbf{u}$ is in $\mathbb{R}^2$, then

Observe: We can **normalize** a vector so it has length one. The resulting vector is a called a **unit vector**.

Example: Let $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.

Remark: If $\mathbf{u}$ and $\mathbf{v}$ are both non-zero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \; ||\mathbf{v}|| \cos\theta,$$

Cauchy-Schwarz and Triangle Inequalities: If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then

Other Useful Properties: Let $\mathbf{u}$ and $\mathbf{v}$ be in $\mathbb{R}^n$. Then

1. $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$

2. $\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}-\mathbf{v}||^2$