Chapter 5: Eigenvalues and Eigenvectors

§5.3 – Diagonalization (Continued)

Recall: An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof:

Example: From last class, the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has 3 distinct eigenvalues $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$. Thus, A is diagonalizable.

Examples: Diagonalize the following matrices, if possible.

1. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Theorem: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n.

(c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k, then the vectors in $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is an eigenvector basis for $\mathbb{R}^n$.

Example: Let $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$.