Chapter 3: Determinants

Section 3.2 – Properties of Determinants (Continued)

Warm-Up Example: Let $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 3 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -3 & -5 & 4 & -2 \end{bmatrix}$. Find $\det A$.

Observations: Let A be an $n \times n$ matrix and B be a row echelon form of A. If $A \rightsquigarrow B$ with s row interchanges, then

Moreover, if A is invertible, then

Theorem: An $n \times n$ matrix A is invertible if and only if

Example: $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 5 & -3 \\ -6 & 7 & -7 \end{bmatrix}$

Determinants and Products:

Example: Consider $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$. We have

Theorem: If A and B are $n \times n$ matrices, then

Careful: In general, $\det(A + B) \neq \det(A) + \det(B)$.

Example: Can there exist a 2×2 matrix A such that $A^2 = \begin{bmatrix} 3 & 6 \\ 4 & 5 \end{bmatrix}$?

Determinants and Transposes:

Theorem: For any $n \times n$ matrix A,

Determinants of Inverses:

Let A be an $n \times n$ invertible matrix. Then

True/False? There exist invertible 5×5 matrices A and B such that $BAB^{-1} = -2A$.

Section 3.3 – Cramer's Rule, Volume, and Linear Transformations

Part I - Cramer's Rule:

Goal: To find a closed-formula solution to $A\mathbf{x} = \mathbf{b}$.

The 2×2 **Case:** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be an invertible matrix, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. We know

Cramer's Rule: Let A be an $n \times n$ invertible matrix, $\mathbf{b}$ be in $\mathbb{R}^n$ and $A_i(\mathbf{b})$ be the matrix

$$A_i(\mathbf{b}) = \left[\begin{array}{ccccc} \mathbf{a_1} & \cdots & \mathbf{b} & \cdots & \mathbf{a_n} \end{array} \right].$$

Example: Solve $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.