Chapter 2: Matrix Algebra

Section 2.9 – Dimension and Rank

Goal: To look at (wonderful!) consequences of working with bases.

Part I - Coordinate Systems

Observation: Let H be a subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_p}\}$. Let $\mathbf{x}$ be a vector in H such that

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_p\mathbf{b_p}$$

and

$$\mathbf{x} = d_1\mathbf{b_1} + \cdots + d_p\mathbf{b_p}$$

for scalars $c_1, \ldots, c_p, d_1, \ldots, d_p$. Then

Definition: Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_p}\}$ be a basis for a subspace H. For each $\mathbf{x}$ in H, the **coordinates of $\mathbf{x}$ relative to $\mathcal{B}$** are

Examples:

1. $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ -3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, then what is $\mathbf{x}$?

2. Let $H = \text{Span}\left\{ \mathbf{b_1} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} \right\}$. Note that $\mathbf{b_1}, \mathbf{b_2}$ are linearly independent. Thus, $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ is a basis for H. Let $\mathbf{y} = \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}$. Is $\mathbf{y}$ in H? If so, find $[\mathbf{y}]_{\mathcal{B}}$.

Part II - The Dimension of a Subspace

Fact: If H is a subspace, then it may have multiple different bases. However, every basis has

Definition: The **dimension** of the zero subspace $\{\mathbf{0}\}$ is defined to be 0. The **dimension** of a non-zero subspace H, denoted $\dim H$, is the number of

Examples:

1. $\dim \mathbb{R}^n =$

2. The dimension of a line through the origin $\mathbf{0}$ in $\mathbb{R}^3$ is

3. The dimension of a plane through the origin $\mathbf{0}$ in $\mathbb{R}^3$ is

4. $H = \text{Span}\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ -1 \\ -6 \end{bmatrix} \right\}$

5. $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \text{ are in } \mathbb{R} \right\}$

6. $H = \text{Span}\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix} \right\}$

The Basis Theorem: Let H be a subspace of $\mathbb{R}^n$ and $\dim H = p$. Then

1. Any linearly independent set of exactly p vectors in H

2. Any set of p vectors in H that spans H is

Examples:

1. $\dim \mathbb{R}^2 = 2$. The set $\mathcal{B} = \left\{ \mathbf{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a linearly independent set of

2. $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \text{ are in } \mathbb{R} \right\}$

True/False? $\mathbb{R}^4$ contains a 5-dimensional subspace H.