Chapter 2: Matrix Algebra

Section 2.2 – The Inverse of a Matrix (Continued)

Question: Invertible matrices have nice properties. How do we actually find the inverse?

2×2 **Matrices:**

Definition: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **determinant** of A is

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\det(A) \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $\det(A) = 0$, then A is not invertible.

Examples:

1. Let $A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$.

2. If $A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, then

Question: What about larger matrices?

Definition: An **elementary matrix** is a matrix that is obtained by performing *one* elementary row operation on an identity matrix.

Example:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

are elementary matrices. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

We have

Facts:

1. If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix is EA where E is the $m \times m$ matrix created by performing the same row operation on I_m.
2. Each elementary matrix E is invertible. Moreover, E^{-1} is the elementary matrix of the same type that transforms E back into I_m.

Example: If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, then

Theorem: Let A be an $n \times n$ matrix. Then A is invertible if and only if A is row equivalent to I_n and, in this case, any sequence of elementary row operations that transforms A into I_n also transforms I_n into A^{-1}.

Punch-Line: To find A^{-1} (if it exists):

Example:

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right]$$