Chapter 2: Matrix Algebra

Section 2.2 – The Inverse of a Matrix (Continued)

Recall: Let A be an $n \times n$ matrix. Then A is invertible if and only if $A \rightsquigarrow I_n$. If A^{-1} exists, then

$$[A \quad | \quad I_n] \rightsquigarrow [I_n \quad | \quad A^{-1}].$$

Note:

Example: Find A^{-1} (if it exists) for $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}$.

Section 2.3 – Characterizations of Invertible Matrices

Goal: To "connect the dots" from our previous work!

The Invertible Matrix Theorem: Let A be an $n \times n$ matrix. The following statements are all equivalent. That is, the statements are either all true or all false.

(a) A is an invertible matrix.

(b) A is row equivalent to

(c) A has

(d) The equation $A\mathbf{x} = \mathbf{0}$ has

(e) The columns of A form a linearly

(f) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is

(g) The equation $A\mathbf{x} = \mathbf{b}$ has

(h) The columns of A

(i) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps

(j) There is an $n \times n$ matrix C such that

(k) There is an $n \times n$ matrix such that

(l) A^T is

Example: Is $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & -1 & 1 \\ 0 & 5 & 5 \end{bmatrix}$ invertible?

Invertible Linear Transformations:

Recall: Let $T : \mathbb{R}^p \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations with standard matrices A and B, respectively.

- Define the composition $S \circ T : \mathbb{R}^p \to \mathbb{R}^m$ by

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$$

  for $\mathbf{x}$ in $\mathbb{R}^p$.

- The standard matrix of $S \circ T$ is

Definition: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **invertible** if there exists a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is invertible. In that case, $S : \mathbb{R}^n \to \mathbb{R}^n$ defined by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is linear and is the unique transformation satisfying

$$S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 2x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix}$. Then T is linear with standard matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}$. In the first example of the day we found

$$A^{-1} = \begin{bmatrix} 2 & -5 & 4 \\ 0 & 1 & -1 \\ -1 & 3 & -2 \end{bmatrix}.$$

So T is invertible and the standard matrix of T^{-1} is A^{-1}. That is,