Chapter 1: Linear Equations in Linear Algebra

Section 1.9 – The Matrix of a Linear Transformation (Continued)

Recall: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $\mathbf{x}$ is in $\mathbb{R}^n$, then

$$T(\mathbf{x}) = x_1 T(\mathbf{e_1}) + \cdots + x_n T(\mathbf{e_n}).$$

That is,

Warm-Up Exercises: Find the standard matrix A for the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$.

1. T is the horizontal shear leaving $\mathbf{e_1}$ unchanged and mapping $\mathbf{e_2} \mapsto \mathbf{e_2} + 3\mathbf{e_1}$.

2. T rotates each point in $\mathbb{R}^2$ about the origin through φ radians, counter-clockwise.

3. T first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

Properties of Transformations: We now investigate special properties of transformations and then focus on these for linear transformations.

Definition: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. T is said to be **onto** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of

2. T is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of

Notes:

1. T is onto $\mathbb{R}^m$ if $T(\mathbf{x}) = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
2. T is 1-1 if $T(\mathbf{x}) = \mathbf{b}$ has a unique solution or no solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

Examples:

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the x_1-axis. This is linear with standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is 1-1 if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

1. T maps *onto* $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
2. T is *1-1* if and only if the columns of A are linearly independent.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the rule

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ -x_1 \\ x_1 + 5x_2 \end{bmatrix}.$$

Show that T is linear. Is T 1-1? Is T onto?