Chapter 1: Linear Equations in Linear Algebra

Section 1.8 – Introduction to Linear Transformations (Continued)

Matrix Transformations: Let A be an $m \times n$ matrix. We can define the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^n$. Note

Examples:

1. Let $A = \begin{bmatrix} 1 & -4 & 2 & -1 \\ -2 & 6 & 4 & -1 \end{bmatrix}$. Define $T : \mathbb{R}^4 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Thus,

   (a) Let $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Find $T(\mathbf{u})$.

(b) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find $\mathbf{x}$ in $\mathbb{R}^4$ such that $T(\mathbf{x}) = \mathbf{b}$. Is $\mathbf{x}$ unique?

2. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Is $\mathbf{y}$ in the range of T?

We now consider special transformations!

Example: Define $T:\mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Definition: A transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if

Examples:

1. Consider $T:\mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = 2\mathbf{x}$.

2. Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then for all $\mathbf{x}, \mathbf{y}$ in R^n and scalars c we have

Properties: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Examples:

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3|x_1| \\ x_1 + x_2 \end{bmatrix}$ is not linear

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through $x_2 = 0$:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}.$$

   Let c, d be scalars and $\mathbf{x}, \mathbf{y}$ be in $\mathbb{R}^2$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

What is $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$?