Chapter 1: Linear Equations in Linear Algebra

Section 1.7 – Linear Independence

Motivating Example: Let $\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$ and let $W = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Can we obtain W with just two of these vectors?

Definition: Let $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be a set of vectors in $\mathbb{R}^n$.

1. $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is said to be **linearly independent** if the equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

2. $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

Example: Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$.

(a) Is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ linearly independent?

(b) If possible, find a dependence relation.

Observe: If $A = [\mathbf{a_1} \ \cdots \ \mathbf{a_n}]$. Then $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{0}.$$

Example: Consider $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ -1 & 4 & -3 \end{bmatrix}$. The system $A\mathbf{x} = \mathbf{0}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 3 & -2 & 0 \\ -1 & 4 & -3 & 0 \end{array}\right]$$

Fact: A set of two vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent if at least one $\mathbf{v_i}$ is a multiple of the other. In other words, $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent if and only if neither of the vectors is

Example: Let $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$. Then

Question: What if we have more than 2 vectors?

Theorem: The set $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ with $p > 2$ is linearly dependent if and only if at least one of the $\mathbf{v_i}$ is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v_1} \neq \mathbf{0}$, then some $\mathbf{v_j}$ with $j > 1$ is a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}$.

Proof: