Chapter 1: Linear Equations in Linear Algebra

Section 1.7 – Linear Independence (Continued)

Warm-Up: Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix}$.

(a) For what value(s) of h is $\mathbf{v_3}$ in Span$\{\mathbf{v_1}, \mathbf{v_2}\}$?

(b) For what value(s) of h is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ linearly dependent?

Food for Thought Example: Last class we saw - if the set $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is linearly dependent with $\mathbf{v_1} \neq \mathbf{0}$, then some $\mathbf{v_j}$ (with $j > 1$) is a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}$. This does not imply that *every* vector in S is a linear combination of the preceding vectors. For example,

$$S = \left\{ \mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \right\}$$

Theorem: If $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a set of vectors in $\mathbb{R}^n$ and $p > n$, then S is linearly dependent.

Theorem: If $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ contains $\mathbf{0}$, then S is linearly dependent.

Examples:

1. Determine if the given set is linearly dependent.

   (a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$

   (c) $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 4/3 \\ 2 \\ 1 \end{bmatrix}$

2. In the matrix $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, note that $\mathbf{a_1} + 2\mathbf{a_2} = \mathbf{a_3}$. Find a non-trivial solution to $A\mathbf{x} = \mathbf{0}$.

Section 1.8 – Introduction to Linear Transformations

Goal: To interpret $A\mathbf{x}$ as A "acting on" $\mathbf{x}$.

Definition: A **transformation** (or **function** of **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Example: Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by the rule

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 + 3 \end{bmatrix}.$$