Chapter 1: Linear Equations in Linear Algebra

Section 1.5 – Solution Sets of Linear Systems

Goal: Write a solution of a linear system as a sum of a particular solution and another vector coming from a special linear system.

Definition: A system of linear equations is called **homogeneous** if it can be written in the form

Notes:

1. Homogeneous systems always have the *trivial solution* $\mathbf{x} = \mathbf{0}$ since

2. $A\mathbf{x} = \mathbf{0}$ has a *non-trivial solution* (i.e., $\mathbf{x} \neq \mathbf{0}$) if and only if there is at least

Examples:

1. The system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

is homogeneous. If $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, then the system is of the form

2. The system $5x_1 - 3x_2 + x_3 = 0$ is homogeneous.

3. The system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \end{aligned}$$

is <u>not</u> homogeneous. Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

4. The system $A\mathbf{x} = \mathbf{0}$ with $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 3 & -1 & 5 & 4 \end{bmatrix}$ has non-trivial solutions. Note that

$$A = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 2 & 1 & 5 & 1 & 0 \\ 3 & -1 & 5 & 4 & 0 \end{array}\right]$$

Theorem: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then then solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v_h}$$

where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Proof that $\mathbf{w}$ *is a solution to* $A\mathbf{x} = \mathbf{b}$*:*