Chapter 1: Linear Equations in Linear Algebra

Section 1.4 – The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Goal: Explore linear combinations with matrix/vector products.

Definition: If A is an $m \times n$ matrix with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$ and $\mathbf{x}$ is in $\mathbb{R}^n$, then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a_1} & \cdots & \mathbf{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

Note: The ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Examples:

1. $\begin{bmatrix} 5 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} =$

2. $\begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} =$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$

Properties of $A\mathbf{x}$: If A is an $m \times n$ matrix and $\mathbf{u}, \mathbf{v}$ are in $\mathbb{R}^n$ and c is a scalar, then

1. $A(\mathbf{u} + \mathbf{v}) =$

2. $A(c\mathbf{u}) =$

Example:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) =$$

We now make connections with linear systems!

Example: The linear system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ x_2 + 8x_3 &= 1 \end{aligned}$$

has the same solution set as

Theorem: If A is an $m \times n$ matrix with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$ and if $\mathbf{b}$ is in $\mathbb{R}^m$, then

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as

Fact: $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A. That is, $A\mathbf{x} = \mathbf{b}$ has a solution if and only if

Question: What if we want $\mathbf{b}$ is in the span of the columns of A for <u>all</u> possible $\mathbf{b}$?

Example: Let $A = \begin{bmatrix} 2 & -1 \\ -10 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be in $\mathbb{R}^2$. Is $A\mathbf{x} = \mathbf{b}$ consistent for <u>all</u> possible $\mathbf{b}$?

Definition: A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^m$ **spans** $\mathbb{R}^m$ if <u>every</u> vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_p}$.

Theorem: Let A be an $m \times n$ matrix. The following are equivalent:

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

Example: $\mathbf{u_1} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ do not span $\mathbb{R}^2$ since $\begin{bmatrix} 2 & -1 \\ -10 & 5 \end{bmatrix}$