Chapter 1: Linear Equations in Linear Algebra

Section 1.3 – Vector Equations

Definitions: Fix a positive integer n.

1. A **vector** (in $\mathbb{R}^n$) is an <u>ordered</u> n-tuple of real numbers

2. $\mathbb{R}^n$

3. $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the

4. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be in $\mathbb{R}^n$ and c be a real number.

   - The **sum** of $\mathbf{x}$ and $\mathbf{y}$ is

   - The **scalar multiple** of $\mathbf{x}$ by c is

Examples:

1. Consider $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$. Then

2. Consider $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$. Then

Properties of $\mathbb{R}^n$: For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^n$ and all scalars c, d in $\mathbb{R}$, we have:

1. $\mathbf{x} + \mathbf{y} =$

2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} =$

3. $\mathbf{x} + \mathbf{0} =$

4. $\mathbf{x} - \mathbf{x} =$

5. $c(\mathbf{x} + \mathbf{y}) =$

6. $(c + d)\mathbf{x} =$

7. $c(d\mathbf{x}) =$

8. $1\mathbf{x} =$

Linear Combinations and Spanning Vectors

Goal: To connect vectors and linear systems of equations.

Definition: Given vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$ and scalars $c_1, \ldots, c_p$ in $\mathbb{R}$, the vector $\mathbf{y}$ given by

$$\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}$$

is called a

Example: Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. Is $\mathbf{b}$ a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$?

Fact: $\mathbf{b}$ is a linear combination of $\mathbf{a_1}, \ldots, \mathbf{a_n}$ if and only if there is a

Definition: Let $\mathbf{v_1}, \ldots, \mathbf{v_p}$ be vectors in $\mathbb{R}^n$. The subset of $\mathbb{R}^n$ **spanned** (or **generated**) by $\mathbf{v_1}, \ldots, \mathbf{v_p}$ is

Examples:

1.

2. Is $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix}$ in $\text{Span}\left\{ \mathbf{u_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$?

3. Is $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 5 \end{bmatrix}$ in $\text{Span}\left\{ \mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ -6 \end{bmatrix} \right\}$?

4. Let $\mathbf{v}$ and $\mathbf{w}$ be non-zero vectors in $\mathbb{R}^3$ where $\mathbf{w}$ is not a scalar multiple of $\mathbf{v}$. Then:
   - $\text{Span}\{\mathbf{v}\}$ is the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$;
   - $\text{Span}\{\mathbf{v}, \mathbf{w}\}$ is the plane in $\mathbb{R}^3$ containing $\mathbf{v}, \mathbf{w}, \mathbf{0}$.