Chapter 1: Linear Equations in Linear Algebra

Section 1.2 – Row Reduction and Echelon Forms (Continued)

Note: A matrix can be row reduced to *many* matrices that are in row echelon form. However, ...

Theorem: Each matrix is row equivalent to *one and only one* reduced (row) echelon matrix.

The Row Reduction Algorithm: This algorithm is used to find the reduced row echelon form of a matrix. We begin with two preliminary definitions.

Definitions: Let A be a matrix.

1. A **pivot position** in A is a

2. A **pivot column** of A is a column that

The Row Reduction Algorithm Via An Example: Our goal is to find $RREF(A)$ where

$$A = \begin{bmatrix} -2 & 4 & 5 & -5 & 3 \\ 1 & -2 & -1 & 3 & 0 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix}.$$

- **Step 1:** Begin with the left-most non-zero column. This is a pivot column. The pivot position is at the top.

- **Step 2:** Select a non-zero entry in the pivot column as a pivot. Interchange rows (if need be) to move this entry into the pivot position.

- **Step 3:** Use row replacements to create zeros in all positions below the pivot.

- **Step 4:** Cover the row with the pivot position as well as all rows above. Apply Steps 1–3 on the remaining submatrix. Repeat this process until there are no non-zero rows to modify. The resulting matrix is in

- **Step 5:** Beginning with the right-most pivot and working upward and to the left, create zeros above each pivot. Also, scale the pivots to 1. The resulting matrix is the

Parametric Descriptions of Solution Sets: We now fine-tune finding the general solution to a linear system of equations.

Definitions: When solving a linear system, variables corresponding to pivot columns are called

Examples: Find the general solution to each of the following linear systems.

1.

$$\begin{aligned} x_1 - 2x_2 + 3x_4 &= -3 \\ 2x_1 + x_2 + 5x_3 + x_4 &= 4 \\ 3x_1 - x_2 + 5x_3 + 4x_4 &= 1 \end{aligned}$$

2.

$$\begin{aligned} x_1 - 3x_2 - x_4 &= -2 \\ x_2 - 4x_5 &= 1 \\ x_4 + 9x_5 &= 4 \end{aligned}$$

The *existence* and *uniqueness* of a linear system can be generalized from these examples in the following theorem:

Theorem: A linear system is consistent if and only if the right-most column of the augmented matrix is *not* a pivot column (i.e., no row of the RREF is of the form

$$[0 \quad \cdots \quad 0 \quad b]$$

where $b \neq 0$). If a linear system is consistent then there is: