Chapter 1: Linear Equations in Linear Algebra

Section 1.1 – Systems of Linear Equations

We begin our investigation of linear systems of equations by setting up the standard terminology.

Definitions:

1. A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

   where

2. A **system of linear equations** (or **linear system**) is a

3. A **solution** to a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when

4. The **solution set** of a linear system is the set of all possible solutions.

5. Two linear systems are called **equivalent** if they have the same

Examples:

1. $5x_1 - 6x_2 + 3 = x_3$ and $x_2 = 2\sqrt{6} - 3x_1 + x_3$ are

2. $x_1x_3 + x_4 = 6$ and $\sqrt{x_1} + 5 = x_2$ are

3.

$$\begin{aligned} 5x_1 - 6x_2 - x_3 &= -3 \\ 3x_1 + x_2 - x_3 &= 2\sqrt{6} \\ x_1 - x_3 &= 10 \end{aligned}$$

is a

4. $(1, 0, -1)$ is a solution to the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Question: How large can a solution set be?

Answer: Think of lines in the plane!

$$\begin{aligned} \ell_1 &: a_1x_1 + a_2x_2 = b_1 \\ \ell_2 &: \widetilde{a_1}x_1 + \widetilde{a_2}x_2 = \widetilde{b_2} \end{aligned}$$

We have three possibilities:

- Case 1: ℓ_1 and ℓ_2 intersect in *one* point

- Case 2: ℓ_1 and ℓ_2 coincide

- Case 3: ℓ_1 and ℓ_2 are parallel

The situation seen with lines is true more generally:

General Fact: A system of linear equations has:

1. no solution, OR
2. exactly one (unique) solution, OR
3. infinitely many solutions.

Definitions: A linear system is called

1. **consistent** if it has

2. **inconsistent** if it has

Question: Given a linear system, how do we find an equivalent linear system which is easier to solve?

Answer: Use matrices! These provide a compact book-keeping tool.

Definition: A **matrix of size** $m \times n$ is a

Example/Terminology: The linear system

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$$

has **coefficient matrix**

and **augmented matrix**

Observe: When we solve a linear system we replace it with an equivalent one which is easier to solve. To find the replacement we only need to keep track of coefficients in a systematic way. We can do **3** valid operations only to the augmented matrix when doing this:

Elementary Row Operations of a Matrix:

1. **Replacement:**

2. **Interchange:**

3. **Scaling:**

Definition: Two matrices are called **row equivalent** if there is a sequence of *elementary row operations* that transforms one matrix into the other.