(11) Let $X = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$. Set

$A = \{(x,y,z) \in X \mid x^2+y^2+z^2 = 1 \text{ and } z > 0\}$ and

$B = \{(x,y,z) \in X \mid x^2+y^2+z^2 = 1 \text{ and } z < 0\}$.

Define $f: A \to B$ by $f(x,y,z) = (x,y,-z)$, and show that $X_f$ is homeomorphic to $S^3 = \{(x,y,z,u) \in \mathbb{R}^4 \mid x^2+y^2+z^2+u^2 = 1\}$.

Solution:

We begin with a lemma:

Lemma: Let $B_i$, $i=1,2$ denote two copies of the 3-dimensional unit ball:

$$B = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\} \text{ (this is } X \text{ renamed)}.$$

Then $S^3 = B_1 \cup_\varphi B_2$, where $\varphi: \partial B_1 \to \partial B_2$ is the identity map $\varphi(x,y,z) = (x,y,z)$.

Proof:

Define $p: S^3 \longrightarrow B^3$ by $p(x,y,z,u) = (x,y,z)$.

Denote the restrictions to

$$S^3_\pm = \{(x,y,z,u) \in S^3 \mid u \geq 0\} \text{ or}$$
$$\{(x,y,z,u) \in S^3 \mid u \leq 0\}.$$

by $p_\pm : S^3_\pm \longrightarrow B^3$. Then each of $p_+$ and $p_-$ is a homeomorphism with $B^3$, since

$$p_\pm^{-1}(x,y,z) = \left(x, y, z, \pm\sqrt{1-x^2-y^2-z^2}\right)$$

defines a continuous inverse $p_\pm^{-1} : B^3 \longrightarrow S^3_\pm$.

(Note $p_\pm$ and $p_\pm^{-1}$ are all continuous maps since they are continuous in every coordinate).

Thus $S^3 \cong B_1^3 \cup_\varphi B_2^3$, where it remains to determin the identification of $\partial B_1^3$ and $\partial B_2^3$. The gluing map is given by

$$\varphi : \partial B_1^3 \xrightarrow{p_+^{-1}} S^3_+ \cap S^3_- \xrightarrow{p_-} \partial B_2^3$$

$$S^3_+ \cap S^3_- \cong \{(x,y,z,u) \in S^3 \mid u=0\} \cong \{(x,y,z) \mid x^2+y^2+z^2=$$

$$\cong \partial B^3.$$

The formula for $\varphi$ is

$$\varphi(x,y,z) = p_-(p_+^{-1}(x,y,z)) = p_-\left(x,y,z,\sqrt{1-x^2-y^2-z^2}\right)$$

$$= p_-(x,y,z,0) = (x,y,z)$$

So the lemma is proved.

Claim: Divide the ball $B^3$ into hemispheres

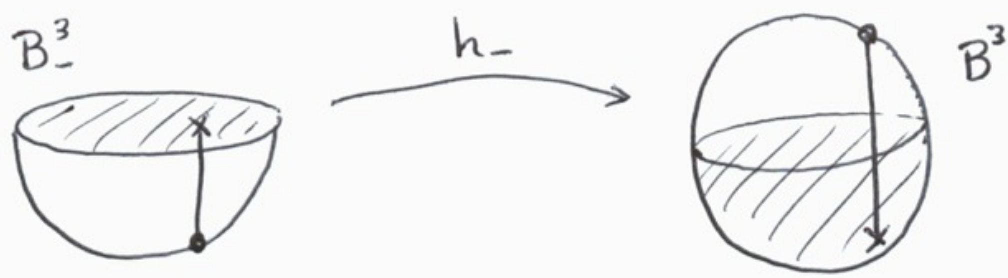
$$B^3_- = \{(x,y,z) \in B^3 \mid z \leq 0\}$$

$$B^3_+ = \{(x,y,z) \in B^3 \mid z \geq 0\}$$

Then each hemisphere is homeomorphic to a copy of $B^3$ via a homeomorphism transforming

$$X_f = B^3_- \oplus B^3_+ / \sim \quad \text{into} \quad S^3 = B^3 \cup_\varphi B^3.$$

Proof: Idea:


Given $(x,y)$ satisfying $x^2+y^2 \leq 1$, define $h_+^{x,y}$ to be any homeomorphism with

$$h_+^{x,y} : [0, \sqrt{1-x^2-y^2}] \longrightarrow [-\sqrt{1-x^2-y^2}, \sqrt{1-x^2-y^2}]$$

and $h_+^{x,y}(0) = -\sqrt{1-x^2-y^2}$, $h_+^{x,y}(\sqrt{1-x^2-y^2}) = \sqrt{1-x^2-y^2}$.

Define $h_-^{x,y}$ to be any homeomorphism

$$h_-^{x,y} : [-\sqrt{1-x^2-y^2}, 0] \longrightarrow [-\sqrt{1-x^2-y^2}, \sqrt{1-x^2-y^2}]$$

satisfying $h_-^{x,y}(0) = -\sqrt{1-x^2-y^2}$, $h_-^{x,y}(-\sqrt{1-x^2-y^2}) = \sqrt{1-x^2-y^2}$.

Define

$$h : X_f \simeq B_-^3 \oplus B_+^3 /_{\sim} \longrightarrow B^3 \cup_\varphi B^3 \simeq S^3$$

by

$$h(x,y,z) = \begin{cases} (x, y, h_-^{x,y}(z)) & \text{if } (x,y,z) \in B_-^3 \\ (x,y,h_+^{x,y}(z)) & \text{if } (x,y,z) \in B_+^3. \end{cases}$$

Then by construction, $h(x,y,z)$ is well defined on equivalence classes.

Question 8: Show that if $\mathbb{R}^2$ is homeomorphic to a CW complex $Y$ of dimension 2, then there is a subcomplex $Y$ that is homeomorphic to $\mathbb{R}$.

Solution: Here is a counterexample to the claim. We will provide $Y$ homeomorphic to $\mathbb{R}^2$ by describing the image of $Y$ under a homeomorphism $h: Y \longrightarrow \mathbb{R}^2$.

The image $h(X^0)$ is the set $\{(n,0) \mid n \in \mathbb{N}^+\}$.

For each $i \in \mathbb{N}^+$, there are two 1-cells $e_i$ and $e_i^{circ}$. The image of $e_i$ under $h$ is:

$$h(e_i) = \{(x,0) \mid x \in [i, i+1]\}$$

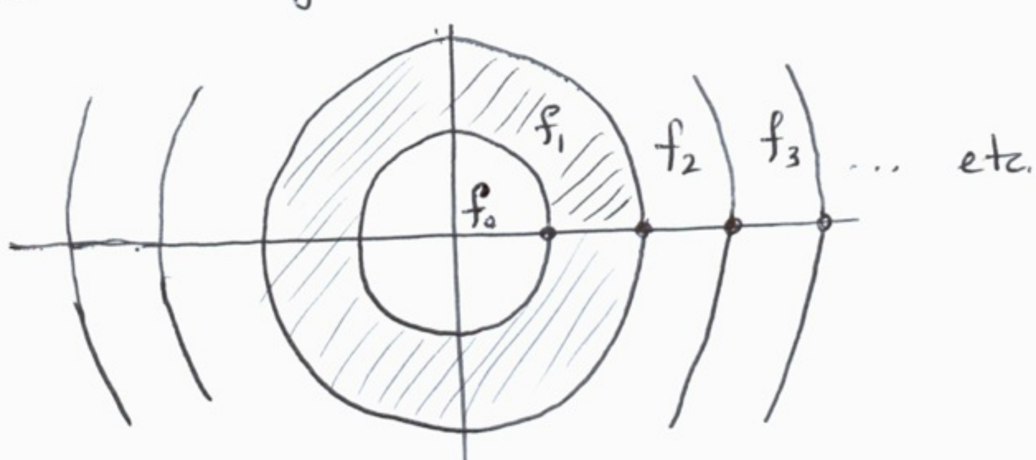
while $h(e_i^{circ}) = \{(x,y) \mid x^2+y^2 = i^2\}$.

There is one 2-cell $f_i$ for each $i \in \mathbb{N}$. The image under $h$ is:

$$h(f_0) = \{(x,y) \mid x^2+y^2 \leq 1\}.$$

$$\text{if } i \geq 1,\ h(f_i) = \{(x,y) \mid i \leq x^2+y^2 \leq i+1\}.$$

So the image of the 1-skeleton under $h$ is



Now suppose $Y$ contains a subcomplex homeomorphic to $\mathbb{R}$, say $Y' \subset Y$. Then $Y' \subset X^1$ is a union of 0-cells and 1-cells. Let $n$ be the smallest integer such that $(n,0) \in h(Y')$.

<u>Claim</u>: There is no neighbourhood of $h^{-1}(n,0)$ homeomorphic to an interval.

<u>Proof</u>: Since $(n,0)$ is the smallest in $h(Y')$, $Y'$ cannot contain the 1-cell $e_{n-1}$, otherwise $h(Y')$ would contain $(n-1,0)$.

Similarly, $Y'$ cannot contain the 1-cell $e_n^{circ.}$ If it did, then removing any point $x \in \mathrm{int}(e_n^{circ})$ from $Y$ would leave a connected set $Y' \setminus \{x\}$, meaning that $Y'$ cannot be homeomorphic to $\mathbb{R}$ ($\mathbb{R} \setminus \{x\}$ is disconnected for all $x$).

Thus the only 1-cell attached to $h^{-1}(n,0)$ is $e_n$, and since $h^{-1}(n,0) \in \partial e_n$ there is no neighbourhood homeomorphic to an open interval. □

⑦ Show that $\mathbb{R}^n$ is a CW complex of dim $n$.

<u>Lemma</u>: For any $m>0$,
$$\partial[0,1]^m = \partial[0,1]\times[0,1]\times\dots\times[0,1] \cup [0,1]\times\partial[0,1]\times\dots\times[0,1] \cup \dots \cup [0,1]\times[0,1]\times\dots\times\partial[0,1].$$

<u>Proof</u>: If $(x_1,\dots,x_m)\in\partial[0,1]^m$, then there exists $i$ such that $x_i = 0$ or $1$. Then
$$(x_1,\dots,x_m)\in[0,1]\times\dots\times\underset{\uparrow}{\partial[0,1]}\times\dots\times[0,1]$$
ith position.

---

<u>Notation</u>: For a product of sets $\prod_{i=1}^n U_i$, let $\partial^j(\prod_{i=1}^n U_i)$ denote
$$U_1\times U_2\times\dots\times\underset{\uparrow}{\partial U_j}\times\dots\times U_n$$
$\partial$ in $j^{th}$ coordinate.

So with this notation, $\partial[0,1]^m = \bigcup_{j=1}^m \partial^j([0,1]^m)$.

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To present $\mathbb{R}^n$ as a CW complex, we will describe the images of all cells $[0,1]^m$ in $\mathbb{R}^n$, $m<n$, as follows: Given $\vec{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n$, consider the set $S_{\vec{k}}$ of all products of the form $\prod_{i=1}^n I_{k_i}$ where $I_{k_i}=[k_i,k_i+1]$ or $I_{k_i}=\{k_i\}$ for $i=1,\dots,n$.

Then the collection of all (images of) cells in $\mathbb{R}^n$ is $C = \bigcup_{\vec{k} \in \mathbb{Z}^n} S_{\vec{k}}$.

First note that these cells cover $\mathbb{R}^n$: Given $(x_1, \dots, x_n) \in \mathbb{R}^n$, we have

$(x_1, \dots, x_n) \in [\lfloor x_1 \rfloor, \lfloor x_1 \rfloor + 1] \times [\lfloor x_2 \rfloor, \lfloor x_2 \rfloor + 1] \times \dots \times [\lfloor x_n \rfloor, \lfloor x_n \rfloor + 1]$

where the latter set is an element of $S_{\vec{k}}$,

$\vec{k} = (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$, hence it is in the collection $C$ of (images of) cells.

Next we check that the boundary of each cell is mapped into a union of cells of lower dimension.

Consider an arbitrary cell $[0,1]^m$ whose image in $\mathbb{R}^n$ $(m<n)$ is a product of the form $\prod_{i=1}^{n} I_{k_i}$ as above, where $I_{k_i} = [k_i, k_i+1]$ for $i \in M \subset \{1, \dots, n\}$ and $I_{k_i} = \{k_i\}$ otherwise $(|M| = m)$. Then the image of the boundary $\partial [0,1]^m = \bigcup_{j=1}^{m} \partial^j [0,1]^m$ is the union $\bigcup_{j \in M} \partial^j \left( \prod_{i=1}^{n} I_{k_i} \right)$. Note that each of

$\partial^j \left( \prod_{i=1}^{n} I_{k_i} \right)$ is a union of two cells of dimension

$m-1$, one for each endpoint of the interval $I_{k_j}$.
Thus the image of the boundary of each $m$-cell lies in the $(m-1)$-skeleton.

(Note: This is just a decomposition of $\mathbb{R}^n$ into $n$-cubes, since (as we saw in the 'covering' argument) every unit $n$-cube with integer coordinates is a cell.

# Topology 3240 March 13

## Ch 7: Compactness.

Def: If $Y \subset X$ are spaces, then a cover of $Y$ is a collection of open sets $\mathcal{O}$ (open in $X$) such that $Y \subset \bigcup_{U \in \mathcal{O}} U$. A subcover of $\mathcal{O}$ is a subset $\mathcal{O}' \subset \mathcal{O}$ such that $\mathcal{O}'$ is a cover of $Y$. A space $Y$ is compact if every open cover has a finite subcover.

Examples:

Theorem: A subset $A \subseteq \mathbb{R}$ is closed and bounded if and only if it is compact.

Proposition: Suppose that $A \subset X$ is compact and $f: X \longrightarrow Y$ is continuous. Then $f(A)$ is compact.

Proof: Let $\{U_i\}_{i \in I}$ be an open cover of $f(A)$. Then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of $A$, so there's a finite subcover $\{f^{-1}(U_i)\}_{i=1}^{n}$. But then

$$f(A) \subset \bigcup_{i=1}^{n} f(f^{-1}(U_i)) = \bigcup_{i=1}^{n} U_i,$$

so $f(A)$ has finite subcover $\{U_i\}_{i=1}^{n}$.

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Changing open to closed and unions to intersections, we also get an equivalent formulation using De Morg

Laws:

A family $\mathcal{F} = \{F_i\}_{i \in I}$ has the finite intersection property if every finite intersection of sets in $\mathcal{F}$ is nonempty.

Proposition: A space $X$ is compact iff for every family $\mathcal{F}$ of closed subsets of $X$, we have:

$\mathcal{F}$ has the finite intersection property $\Longrightarrow \bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

Solution: Apply De Morgan's Laws to the definition of compactness.

Related, we have

Proposition 6: A space $X$ is compact iff for every family $\mathcal{F}$ of subsets having the finite intersection property we have $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$.

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§7.2 Properties of compact spaces.

Proposition: If $X$ is compact and $A \subset X$ is closed, then $A$ is compact.

Proof: Let $\{U_i\}_{i \in I}$ be an open covering of $A$. Then $\{U_i\}_{i \in I} \cup A^c$ is an open covering of $X$, so choose a finite subcover $\mathcal{U}$ of $X$. Then $\mathcal{U} \setminus \{A^c\}$ is a finite subcover of $A$.

<u>Proposition</u>: Every compact subset of a Hausdorff space is closed.

<u>Proof</u>: We show that if $A \subset X$ is ~~closed~~ compact, $X$ Hausdorff, then $\forall x \in A^c$, $x \in \operatorname{int}(A^c)$.

Given $x \in A^c$, for every $y \in A$ there are nbhds $U_x^y$ of $x$ and $V_y$ of $y$ such that $U_x^y \cap V_y = \emptyset$. Then $\{V_y\}_{y \in A}$ is an open covering of $A$, so there is a finite subcover $\{V_1, \ldots, V_n\}$ that still covers $A$. Each of $V_1, \ldots, V_n$ has an associated neighbourhood $U_i$ of $x$. Then $\bigcap_{i=1}^{n} U_i$ is again an open nbhd of $x$ (since the intersection is finite), and $\left(\bigcap_{i=1}^{n} U_i\right) \cap A = \emptyset$ since the $V_i$ cover $A$ and $U_i \cap V_i = \emptyset$. Thus $x \in \operatorname{int}(A^c)$.

<u>Corollary</u>: Suppose that $X$ is compact and $Y$ is Hausdorff. If $f: X \to Y$ is continuous and bijective, then $f$ is a homeomorphism.

<u>Proof</u>: Given $U \subset X$ open, we must show that $f(U)$ is open. However, this follows from:

$U$ open $\Rightarrow U^c$ closed

$\Rightarrow U^c$ compact since $X$ compact

$\Rightarrow f(U^c)$ compact

$\Rightarrow f(U^c)$ closed, since $Y$ Hausdorff

$\Rightarrow f(U)$ open. //

## Theorem (Proof wiki)

Let $X_1$ and $X_2$ be spaces. Then $X_1 \times X_2$ is compact iff $X_1$ and $X_2$ are compact.

<u>Proof</u>: ($\Rightarrow$) If $X_1 \times X_2$ is compact then $p_i : X_1 \times X_2 \to X_i$ provides a surjection from $X_1 \times X_2$ onto $X_i$, $i = 1, 2$. Since $p_i$ is continuous and $X_1 \times X_2$ is compact, $X_i$ is compact also.

($\Leftarrow$)

Suppose $X_1$ and $X_2$ are compact. Let $\mathcal{W}$ be an open covering of $X_1 \times X_2$. Define the terminology <u>good</u> as follows:

A subset $A \subset X_1$ will be called good for $\mathcal{W}$ if $A \times X_2$ is covered by a finite subset of $\mathcal{W}$. We'll show $X_1$ is good for $\mathcal{W}$.

We first show that $X_1$ is <u>locally good</u>, i.e. $\forall x \in X_1$ $\exists$ an open set $U(x)$ such that $x \in U(x)$ and $U(x)$ is good.

Fix $x \in X_1$. For each $y \in X_2$, $(x, y) \in W(y)$ for some $W(y) \in \mathcal{W}$. There exists a basic open set containing $(x, y)$ that lies entirely in $W(y)$, ie $\exists$ $U(y)$, $V(y)$ open in $X_1$, $X_2$ s.t.

$$(x, y) \in U(y) \times V(y) \subseteq W(y).$$

Then $\{V(y) \mid y \in X_2\}$ is an open cover of $X_2$, choose a finite subcover $\{V(y_1), \dots, V(y_r)\}$ and set $U(x) = U(y_1) \cap \dots \cap U(y_r)$. Then

$$U(x) \times V(y_i) \subset U(y_i) \times V(y_i) \subseteq W(y_i)$$

Therefore

$$U(x) \times X_2 = U(x) \times \bigcup_{i=1}^{r} V(y_i) \subseteq \bigcup_{i=1}^{r} W(y_i),$$ so $U(x)$ is good, ie $X_1$ is locally good.

Now we remark: If $A_1, \dots, A_r$ are all good subsets of $X_1$, then so is $A = \bigcup_{i=1}^{r} A_i$. For each $A_i \times X_2$ is covered by a finite subcover $\mathcal{W}_i$ of $\mathcal{W}$, hence $A \times X_2 = \bigcup_{i=1}^{r} (A_i \times X_2)$ is covered by $\bigcup_{i=1}^{r} \mathcal{W}_i$, which is a finite subset of $\mathcal{W}$.

Now use localgoodness + unions to show $X_1$ is good. The sets $\{U(x) \mid x \in X_1 \text{ and } U(x) \text{ is good}\}$ is an open covering of $X_1$ since $X_1$ is locally good. There is a finite subcover since $X_1$ is compact.