MATH 3240 March 4

Let $I = [0,1]$. A <u>path</u> in a space X is a continuous map $\alpha: I \longrightarrow X$.

<u>Def</u>: A space X is <u>path connected</u> if for every two points $x, y \in X$ $\exists$ a path α with $\alpha(0) = x$ and $\alpha(1) = y$.

<u>Remark</u>. We don't need to use I in this definition, it could be any interval $[a,b] \subset \mathbb{R}$, since $[a,b] \cong [0,1]$.

<u>Example</u>: $\mathbb{R}^n$ is connected, because $\vec{v}, \vec{w} \in \mathbb{R}^n$ are connected by a straight line:

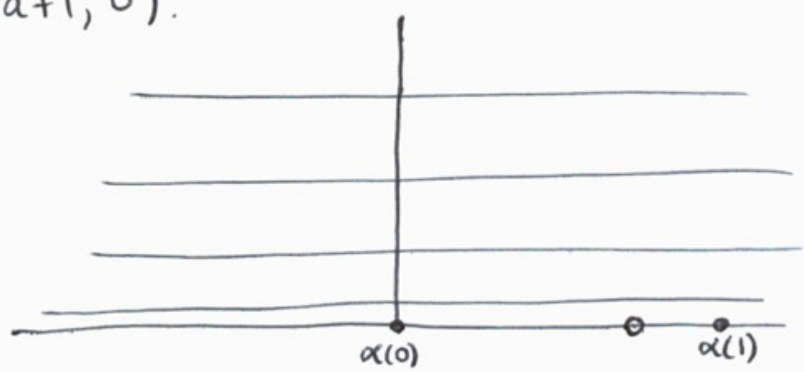
$$\alpha(t) = \vec{v} + t(\vec{w} - \vec{v}), \quad \alpha(0) = \vec{v}, \quad \alpha(1) = \vec{w}.$$

<u>Example</u>: Recall the space

$$X = \{(x,y) \mid y = \tfrac{1}{n}, n \in \mathbb{N}\} \cup \{(0,y) \mid y \geq 0\} \cup \{(x,0) \mid x \in \mathbb{R}\}.$$

we saw last day that removing any subset of $\{(x,0) \mid x \in \mathbb{R}\}$ leaves the space connected, contradicting our intuition.

Set $A = (a,0)$, $a > 0$ $a \in \mathbb{R}$. We'll see that $X \setminus A$ is not path connected, by contradiction. So let $\alpha: I \longrightarrow X \setminus A$ be a path with $\alpha(0) = (0,0)$ and $\alpha(1) = (a+1, 0)$.



We consider two cases.

Case 1) $\alpha(I)$ contains finitely many points $(a, \frac{1}{n})$, $n \in \mathbb{N}$.

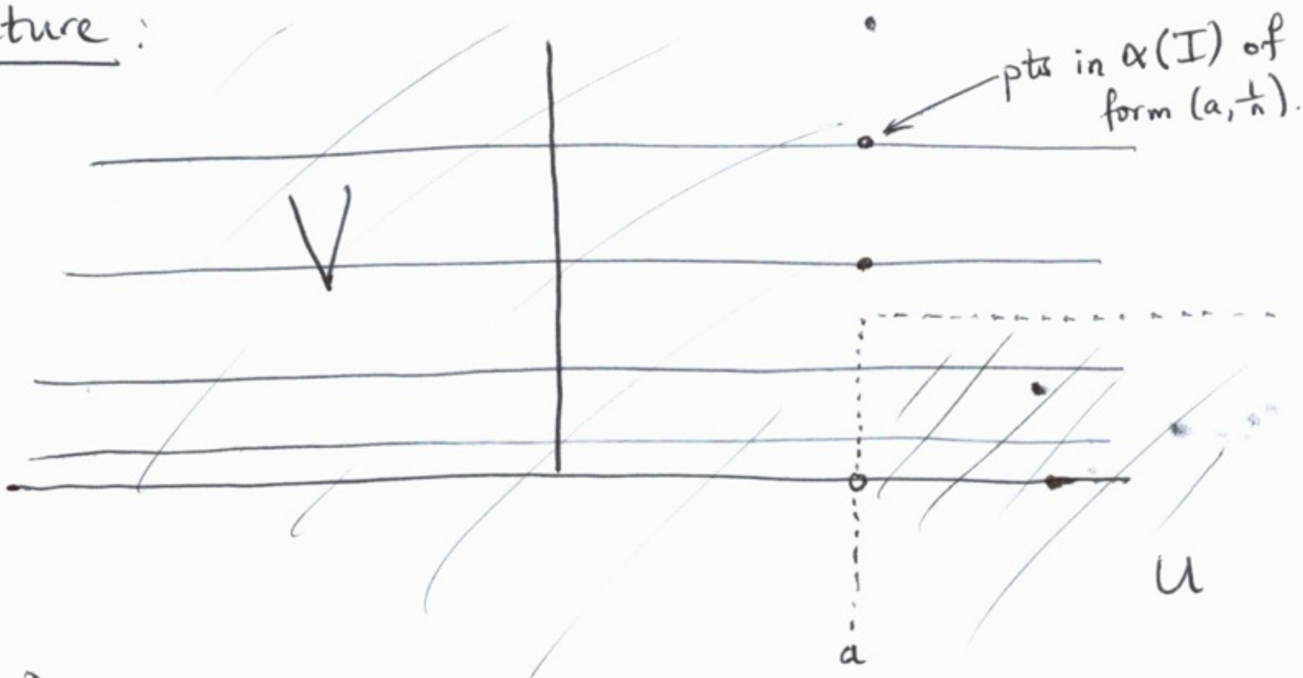
Let $N = \max\{n \in \mathbb{N} \mid (a, \frac{1}{n}) \in \alpha(I)\}$, and $M = \dfrac{\frac{1}{N} + \frac{1}{N+1}}{2}$.

Set $U = (a, \infty) \times (-\infty, M) \subset \mathbb{R}^2$ and $V = \mathrm{int}(U^c)$.

<u>Claim</u>: $\{\alpha^{-1}(U), \alpha^{-1}(V)\}$ separate I.

<u>Proof of claim</u>: Both sets are open, and nonempty since $0 = \alpha^{-1}((0,0)) \in \alpha^{-1}(V)$, and $1 = \alpha^{-1}((a+1, 0)) \in \alpha^{-1}(V)$.
Since $U \cap V = \emptyset$, so too we have $\alpha^{-1}(V) \cap \alpha^{-1}(U) = \emptyset$.
Last, since $X \setminus (U \cup V) = \{(a, \frac{1}{N+i})\}_{i=1}^{\infty} \cup \{(a, 0)\}$, the image $\alpha(I)$ lies entirely in $U \cup V$. Thus. $\alpha^{-1}(U) \cup \alpha^{-1}(V) = I$; so we have a contradiction.

<u>Picture</u>:



Case 2) $\alpha(I)$ contains infinitely many of the points $(a, \frac{1}{n})$, $n \in \mathbb{N}$.

Then $\{\alpha^{-1}(a, \frac{1}{n})\}_{n \in \mathbb{N}}$ is an infinite subset of I, so it has an accumulation point $x \in I$ by Bolzano-Weierstrass. But then $\alpha(x) \in X$ is an accumulation point of $\{\alpha(\alpha^{-1}(a, \frac{1}{n}))\}_{n \in \mathbb{N}} \subset X$, since α is continuous. However, $\{(a, \frac{1}{n})\}_{n \in \mathbb{N}} \subset X$ has no accumulation points in X, contradiction.

Thus:

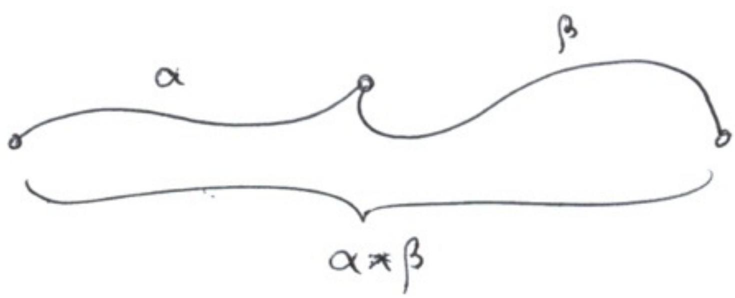
Connectedness does not imply path connectedness.

Proposition: If X is a path-connected topological space, then it is connected.

Proof: Suppose X is not connected, say $\{A, B\}$ is a separation of X. Choose $a \in A$, $b \in B$, and let $\alpha : I \longrightarrow X$ be a path connecting a and b. Then $\{\alpha^{-1}(A), \alpha^{-1}(B)\}$ is a separation of I, contradiction. ∎

Definition: Suppose that $\alpha : I \longrightarrow X$ and $\beta : I \longrightarrow X$ are two paths with $\alpha(1) = \beta(0)$. Define $\alpha * \beta : I \longrightarrow X$ by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

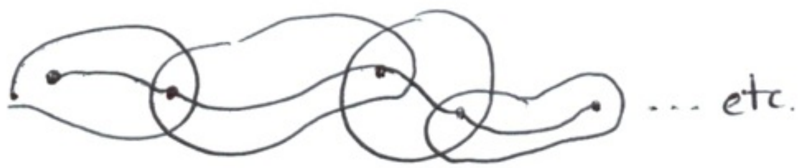


In general, path connectedness is stronger than connectedness. However.

<u>Proposition</u>: Every connected open subset of $\mathbb{R}^n$ is path connected.

<u>Lemma</u>: Let $A_i \subset X$ be path connected subsets for $i \in \mathbb{N}$. If $A_i \cap A_{i+1} \neq \emptyset$ $\forall i$, then $\bigcup_{i=1}^{\infty} A_i$ is path connected.

<u>Proof</u>:



Let $\{a_i\}_{i=1}^{\infty}$ be a family of points with $a_i \in A_i \cap A_{i+1}$ $\forall i$. Let $\alpha_i : I \longrightarrow X$ be a path with $\alpha_i(0) = a_i$, $\alpha_i(1) = a_{i+1}$, and let $x, y \in \bigcup_{i=1}^{\infty} A_i$ be given. Suppose $x \in A_j$ and $y \in A_k$, WLOG $j < k$. Choose $\beta_1, \beta_2 : I \longrightarrow X$ satisfying $\beta_1(0) = x$, $\beta_1(1) = a_j$, $\beta_2(0) = a_k$, $\beta_2(1) = y$. Then

$$\beta_1 * \alpha_j * \alpha_{j+1} * \dots * \alpha_k * \beta_2$$

is a path connecting x and y.

<u>Proof of proposition</u>:

Let $A \subset \mathbb{R}^n$ be a connected subset, and choose $p, q \in A$. Let $\mathcal{O} = \{B \subset \mathbb{R}^n \mid B \text{ is an open ball in } A\}$. Define a sequence of open sets U_i as follows.

Let U_1 be an open ball in $\mathcal{O}$ containing p. Then $\forall i > 1$, set U_i to be the union of balls $B \subset \mathcal{O}$ satisfying $B \cap U_{i-1} \neq \emptyset$.

Set $V = \bigcup_{i=1}^{\infty} U_i$, and suppose $\exists B \in \mathcal{O}$ that is not used in the construction of some U_i. Then the union W of all such balls is open, and $W \cap V = \emptyset$ by construction of the U_i's. Moreover, $W \cup V = A$ since every $B \in \mathcal{O}$ is used in the construction of either V or W, and $\bigcup_{B \in \mathcal{O}} B = A$ since A is open.

Thus $\{V, W\}$ is a separation of A, contradicting connectedness. Thus every $B \in \mathcal{O}$ is used in some U_i. In particular, $q \in V$ and $\exists$ a sequence $\{B_i\}_{i=1}^{n}$ of balls with $B_i \cap B_{i+1} \neq \emptyset$ s.t. $p \in B_1$ and $q \in B_n$. The proof now follows from the lemma.

Terminology

<u>Definition</u>: A path component of a space X is a maximal (with respect to inclusion) path connected subset of X.

<u>Proposition</u>: If a space X is path connected and $f: X \longrightarrow Y$ is continuous, then $f(X)$ is a path connected subset of Y.

<u>Proof</u>: Given $x, y \in f(X)$, choose $p, q \in X$ s.t. $f(p) = x$ and $f(q) = y$. Then choose $\alpha: I \longrightarrow X$ satisfying $\alpha(0) = p$, $\alpha(1) = q$. Then $f \circ \alpha: I \longrightarrow Y$ satisfies $f \circ \alpha(0) = x$ and $f \circ \alpha(1) = y$.

Proposition: If $\{X_i\}_{i\in I}$ are path connected, so is $\prod_{i\in I} X_i$

Proof: Denote the projections by $p_j : \prod_{i\in I} X_i \longrightarrow X_j$.

Choose $\overset{(x_j)}{\overset{\shortparallel}{x}}, \overset{(y_j).}{\overset{\shortparallel}{y}} \in \prod_{i\in I} X_i$. For all $j\in I$, let $\alpha_j : I \longrightarrow X_j$ be a path connecting $p_j(x)$ to $p_j(y)$. Define a new path $\alpha : I \longrightarrow \prod_{i\in I} X_i$ by $\alpha(t) = (\alpha_j(t)) \in \prod_{i\in I} X_i$.

Since each component of α is a continuous map, it is continuous. Moreover, $\alpha(0) = (\alpha_j(0)) = (x_j) = x$ and $\alpha(1) = (\alpha_j(1)) = (y_j) = y$ by construction. Thus $\prod_{i\in I} X_i$ is path connected.

Note: We omit all book material on S-connectedness and S'-connectedness.

Local properties

Connectedness/path connectedness are useful properties, but sometimes it is sufficient if they hold "locally".

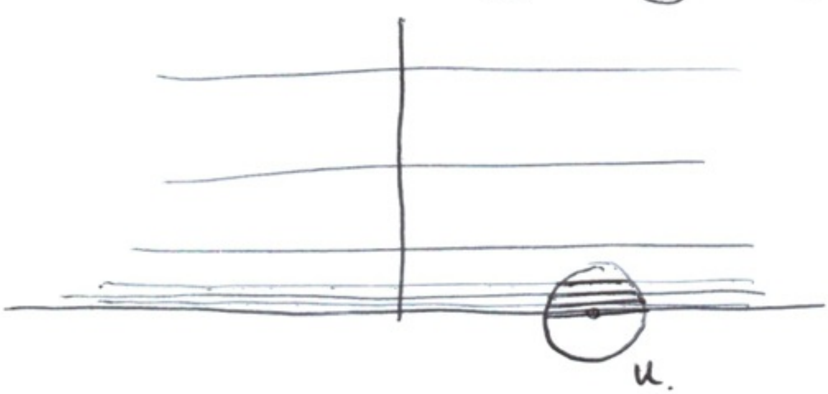
Definition: A space X is locally (path connected) connected of for every neighbourhood U of x, there is a (path) connected nbhd V of x s.t. $x \in V \subset U$.
If X is locally (path) connected at every point, then X is connected.

(connected)
Examples: An interval $(a,b) \subseteq \mathbb{R}$ is connected and locally connected. The subspace $[-1,0) \cup (0,1]$ is not connected, but it is locally connected.

The space

$$X = \{(x, \tfrac{1}{n}) \mid x \in \mathbb{R}, n \in \mathbb{N}\} \cup \{(0,y) \mid y \in \mathbb{R}\} \cup \{(x,0) \mid x \in \mathbb{R}\}$$

is connected, but not locally connected: Given $(a,0) \in X$, $a \neq 0$, any open neighbourhood U satisfying $U \cap \{(0,y) \mid y \in \mathbb{R}\} = \emptyset$ intersects X in infinitely many disjoint segments:



The subspace $\mathbb{Q} \subset \mathbb{R}$ is neither connected nor locally connected.

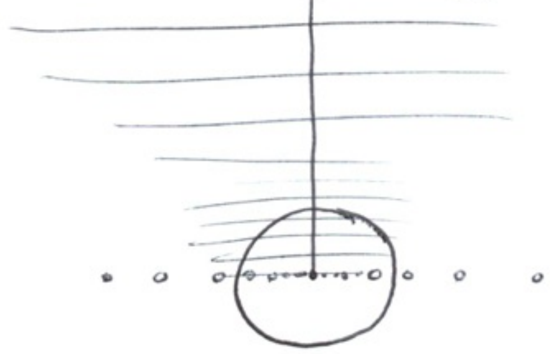
Path connected examples:

The space $\mathbb{R}^n$ is both path connected (straight lines) and locally path connected, since every basis element $(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ is path connected (straight lines suffice).

The space X from the previous example is path connected, but not locally path connected, by the same reasoning. As before, $[-1, 0) \cup (0, 1]$ is not path connected but is locally path connected.

Examples showing local path connectedness and local connectedness are different:

The space $X \setminus \{(\frac{1}{n}, 0) \mid n \in \mathbb{Z}\}$ is locally connected at $(0,0)$, but not locally path connected:



The most important fact about these 'local' concepts is:

<u>Theorem</u>: A connected, locally path connected space is path connected.

<u>Proof</u>: Let X be a locally path connected space which is not path connected. We'll show X is not connected.

Fix $p \in X$. Let C be the set of all points in X that can be joined to p by a path. Since $C \neq \emptyset$ it's enough to show C is clopen.

First, to see open: Let $c \in C$ and choose an open path connected nbhd $\mathcal{U}$ of c. Then $u \in \mathcal{U}$ can be connected to c by a path γ, and c is connected to p by α. Thus $\alpha * \gamma$ connects p to u. Thus $\mathcal{U} \subset C$ and C is open.

To see C is closed: let $c \in \overline{C}$ and choose $\mathcal{U}$ an open path connected nbhd of c. Then $C \cap \mathcal{U} \neq \emptyset$, so choose $u \in C \cap \mathcal{U}$. Join u to p by γ, and c to u by α. Then $\gamma * \alpha$ joins p to c, so $\overline{C} \subset C$ and C is closed.

So $\exists$ a nonempty clopen set in X, C. Unless X is path connected, $C^c \neq \emptyset$. Thus X is not connected.