MATH 3240 Topology 1 Feb 25.

Chapter 6

Connectedness.

Definition. A space X is disconnected if it is a union of disjoint, nonempty open subsets. I.e. $\exists A, B < X$ open s.t. A, B nonempty, $A \cap B = \emptyset$ and $A \cup B = X$.
Otherwise, X is connected.
The sets $\{A, B\}$ are a separation of X.

Example: $\mathbb{R}$ is connected.

Lemma: Let $A \subseteq \mathbb{R}$, A nonempty. If $A' \cap A^c \neq \emptyset$, then A^c is not open.

Proof: Given $x \in A' \cap A^c$, every nbhd U of x contains points of A, and thus $U \not\subset A^c$. But then $x \in A^c$ yet $x \notin \mathrm{int}(A^c)$. So A^c is not open.

Proof that $\mathbb{R}$ is connected: Suppose $\mathbb{R} = A \cup B$, where $A^c = B$ and A, B nonempty and open. We'll show either $A' \cap B \neq \emptyset$ or $B' \cap A \neq \emptyset$.
Choose $a \in A$ and $b \in B$, WLOG suppose $a < b$. Set $X = \{x \in A \mid x < b\}$, and $s = \sup X$. Then $s \in \overline{A} = A \cup A'$.
If $s \notin A$ but $s \in A'$, then $s \in B$ and we're done.
Otherwise $s \in A$ and every point x between s and b lies in B, thus $s \in B'$, and we're done again.

We can generalize this slightly by defining:

<u>Def</u>: An ordered space X is a linear continuum if the linear ordering satisfies

(i) Every subset that is bounded above has a least upper bound

(ii) For every $x, y \in X$ with $x < y$, $\exists\, z$ s.t. $x < z < y$.

<u>Proposition</u>: Every linear continuum is connected.

<u>Example</u>: The Sorgenfrey line is not connected. Recall the topology is generated by sets of the form $[a,b)$ where $a<b$. Then

$$\mathbb{R} = (-\infty, 0) \cup [0, \infty) \text{ is a disjoint union of open sets.}$$

<u>Proposition</u>: If $\{A, B\}$ is a separation of X, then X is homeomorphic to $A \oplus B$ (each equipped with the subspace topology). (Note the converse is trivially true as well)

<u>Proof</u>: Assume $\{A, B\}$ is a separation of X. We prove the topologies on $A \oplus B$ and X are the same, since $X = A \oplus B$ as underlying sets.

So suppose $U \subset X$ is open. Then $U \cap A$ and $U \cap B$ are open, so U is open in $A \oplus B$. On the other hand if $U \subset A \oplus B$ is open, then $U \cap A$ and $U \cap B$ are open by definition of the topology on $A \oplus B$, so

$$(U \cap A) \cup (U \cap B) = U \text{ is open in } X.$$

<u>Theorem</u>: A space X is connected iff the only clopen sets are X and $\emptyset$.

<u>Proof</u>: ($\Rightarrow$) Suppose there exists $A \subset X$ proper, nonempty and clopen. Then A is open and A^c is open, so $X = A \cup A^c$ is not connected.

($\Leftarrow$) Suppose X is disconnected, say $X = A \cup B$. Since A and B are nonempty open sets with $A = B^c$, there are clopen sets aside from X and $\emptyset$.

<u>Note</u>: A subspace $A \subset X$ can be connected (in the subspace topology as well).

<u>Theorem</u>: If $A \subset X$ is connected, then so is $\bar{A}$.

<u>Proof</u>: Suppose A is connected, and let $\bar{A} = B \cup C$ be a decomposition/separation of $\bar{A}$, so $B \cap \bar{C} = C \cap \bar{B} = \emptyset$.

Since $\bar{A}$ is closed, taking closures gives

$\bar{A} = \overline{B \cup C} = \bar{B} \cup \bar{C} = B \cup C$, since B and C ~~are clopen~~ give $\bar{A}$. But now since $B \cap \bar{C} = C \cap \bar{B} = \emptyset$, $\bar{B} \subset B$ and $\bar{C} \subset C$, so B and C are closed.

Thus $A = (B \cap A) \cup (C \cap A)$ is a decomposition into separated open (or closed) sets. Hence either

$B \cap A = \emptyset$ or $C \cap A = \emptyset$

$\Rightarrow A \subset C \cap A$ $\qquad$ $\Rightarrow A \subset B \cap A$

$\Rightarrow \bar{A} \subset C$ $\qquad$ $\Rightarrow \bar{A} \subset B$

$\Rightarrow B = \emptyset$ ↯, $\qquad$ $\Rightarrow C = \emptyset$.

Proposition: a) Suppose $\{A_i\}_{i\in I}$ are connected subspaces of X, and suppose $\bigcap_{i\in I} A_i \neq \emptyset$. Then $\bigcup_{i\in I} A_i$ is connected.

b) Suppose $\{A_i\}_{i\in I}$ are connected subspaces of X, and A_0 is connected and satisfies $A_0 \cap A_i \neq \emptyset$ $\forall i$. Then $A_0 \cup \left(\bigcup_{i\in I} A_i\right)$ is connected.

Proof. (a) Suppose $\bigcup_{i\in I} A_i$ is not connected. Then there exists a nonempty, proper clopen $B \subset \bigcup_{i\in I} A_i$.

Given $j\in I$, if $B\cap A_j \neq \emptyset$ then $A_j \subset B$ since A_j is connected. Thus $B = \bigcup_{j\in J} A_j$ for some subset $J \subset I$.

Since B is proper, $\exists\, k\in I$ such that $A_k \not\subset B$, yet our assumption $\bigcap_{i\in I} A_i \neq \emptyset$ guarantees that $A_k \cap B \neq \emptyset$.

Then $A_k \cap B$ is both open and closed in A_k, contradicting the fact that A_k is connected.

(b) Similar, probably will be on an assignment.

Or set $B_i = A_i \cup A_0$ $\forall i\in I$? .... Then proceed as in (a).

Example: Set

$$X = \bigcup_{n=1}^{\infty} \{(x, \tfrac{1}{n}) \mid x\in\mathbb{R}\} \cup \{(0,y) \mid y\in\mathbb{R}\} \cup \{(x,0) \mid x\in\mathbb{R}\}$$

Then by the previous proposition, X is connected.

On the other hand, $Y = X \setminus \{(0,0)\}$ is still connected.
In fact if $X_0 \subset \{(x,0) \mid x \in \mathbb{R}\}$ then $X \setminus X_0$ is still connected, though intuitively this might disagree with you.
Why connected?
Consider $X \setminus \{(x,0) \mid x \in \mathbb{R}\} \subset X \setminus X_0$. The closure in $X \setminus X_0$ is

$$\overline{X \setminus \{(x,0) \mid x \in \mathbb{R}\}} = X \setminus X_0 .$$

Since $X \setminus \{(x,0) \mid x \in \mathbb{R}\}$ is connected by the previous theorem, so is its closure $X \setminus X_0$.

<u>Example</u>: Consider the infinite product $\prod_{i=1}^{\infty} \{0,1\}$ of $\{0,1\}$ with the discrete topology.
This space is <u>totally</u> <u>disconnected</u>, in the sense that the only connected subsets are singletons.
To see this, let $U \subset \prod_{i=1}^{\infty} \{0,1\}$ and suppose U contains two points $(y_i)_{i=1}^{n}$ and $(x_i)_{i=1}^{n}$. Suppose that they differ in the mth position, say $x_m \neq y_m$, and let $p_m : \prod_{i=1}^{\infty}\{0,1\} \to \{0,1\}$ denote the projection map. Then

$$V_1 = p^{-1}(x_m) \quad \text{and} \quad V_2 = p^{-1}(y_m)$$

are disjoint open sets and

$$V_1 \cap U \,,\ V_2 \cap U \ \text{disconnect } U.$$

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§6.2. Properties of connected spaces.

Perhaps the most useful is that connectedness is preserved by continuous maps.

Proposition: If X is connected and $f: X \to Y$ continuous, then $f(X)$ is connected.

Solution: Suppose $f(X)$ is disconnected, let $\{A, B\}$ be a separation of $f(X)$. Then it is easy to check that $\{f^{-1}(A), f^{-1}(B)\}$ is a separation of X.

Corollary: Connectedness is a topological property.

Example: (IV theorem).

If $f: [a,b] \to \mathbb{R}$ is a continuous function and x is between $f(a)$ and $f(b)$, then $\exists\, c \in [a,b]$ s.t. $f(c) = x$.

Proof: Suppose not. Then $f([a,b])$ is connected, yet if x satisfies $f(a) < x < f(b)$ or $f(b) < x < f(a)$ and $\nexists\, c$ s.t. $f(c) = x$, then $\{(-\infty, x), (x, \infty)\}$ is a separation of $f(X)$. Contradiction.

Corollary. If $f: [a,b] \to [a,b]$ is continuous, then $\exists\, x \in [a,b]$ s.t. $f(x) = x$. (1-dimensional Brouwer fixed point theorem).

Pf: Set $g(x) = f(x) - x$. Then $g(a) \geq 0$ and $g(b) \leq 0$, so $\exists$ c s.t. $g(c) = 0$, by the IV theorem.

Proposition: Suppose that X and Y are topological spaces. Suppose that $\forall x \in X$, $X \setminus \{x\}$ is connected, yet $\exists y \in Y$ s.t. $Y \setminus \{y\}$ is disconnected. Then X and Y are not homeomorphic.

Proof: Let $f: X \longrightarrow Y$ be a homeomorphism, and choose $y \in Y$ s.t. $Y \setminus \{y\}$ is disconnected. Let $\{A, B\}$ be a separation of $Y \setminus \{y\}$. Then $\{f^{-1}(A), f^{-1}(B)\}$ is a separation of $X \setminus \{f^{-1}(y)\}$, which is connected. Contradiction.

Example: The circle $S^1 \cong \{(x,y) \mid x^2+y^2=1\}$ and the bouquet of 2 circles

$$X = \{(x,y) \mid x^2+y^2=1\} \cup \{(x,y) \mid x^2+(y-2)^2=1\}$$

are not homeomorphic: Clearly $S^1 \setminus \{p\}$ is connected $\forall p \in S^1$, while $X \setminus \{(0,1)\}$ is disconnected. The sets $\{(x,y) \mid y>1\} \cap X$ and $\{(x,y) \mid y<1\} \cap X$ provide a separation.

<u>Proposition</u>: Let X and Y be spaces. Suppose that $\forall x \in X$, $X \setminus \{x\}$ is disconnected, and $\exists y \in Y$ s.t. $Y \setminus \{y\}$ is connected. Then X and Y are not homeomorphic.

<u>Proof</u>: Similar.

<u>Example</u>: The spaces $(0,1)$ and $[0,1]$ are not homeomorphic: $(0,1) \setminus \{x\}$ is disconnected $\forall x \in (0,1)$, while $[0,1] \setminus \{0\} = (0,1]$ is connected.

<u>Example</u>: $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic, since $\forall x \in \mathbb{R}$, $\mathbb{R} \setminus \{x\}$ is disconnected while $\mathbb{R}^2 \setminus \{(x,y)\}$ is connected $\forall (x,y) \in \mathbb{R}^2$.

<u>Proposition</u>: A space is connected iff there is no map $f: X \longrightarrow \{0,1\}$ that is continuous and surjective ($\{0,1\}$ has the discrete topology).

<u>Proof</u>: The idea is that if $\{A,B\}$ separate X, then

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

is continuous. Just check the details.

<u>Theorem</u>: Suppose $\{X_i\}_{i \in I}$ are connected. Then $\prod_{i \in I} X_i$ is connected.

Proof: We do the finite case first, so consider two spaces X and Y. Choose a "base point" $(a,b) \in X \times Y$, note that the "horizontal slice" $X \times \{b\}$ is connected if X is connected, and the "vertical slice" $\{a\} \times Y$ is connected if Y is. Therefore the "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected for all $x \in X$, being the union of two connected spaces that have the point (x,b) in common. Now consider

$$\bigcup_{x \in X} T_x ,$$

this union is connected since it is a union of connected spaces having (a,b) in common. The proof for an arbitrary finite product $X_1 \times \dots \times X_n$ follows by induction.

Now, we do the infinite case. Consider $\prod_{i \in I} X_i$, where X_i are connected.

Fix $x = (x_i)_{i \in I}$ in $\prod_{i \in I} X_i$. For every finite subset $T \subset I$, set $C(T) = \prod_{i \in I} A_i$, where $A_i = \{x_i\}$ if $i \notin T$ and $A_i = X_i$ if $i \in T$. Then $C(T)$ is homeomorphic to $\prod_{i \in T} X_i$ and thus is connected by the finite case.

Now since $x \in \bigcap_{\substack{T \subset I \\ \text{finite}}} C(T)$, it follows that $Y = \bigcup_{\substack{T \subset I \\ \text{finite}}} C(T)$

is connected. Then we need the following lemma:

<u>Lemma</u>: Y is a dense subset of $\prod_{i \in I} X_i$.

Then since Y is connected, $\overline{Y} = \prod_{i \in I} X_i$ is connected.

<u>Example</u>: This result does not hold if the product is equipped with other topologies, such as the box topology.

Consider $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$ with the box topology.

Set $A = \{(x_i) \in \mathbb{R}^\omega \mid \{x_i\}_{i=1}^{\infty} \text{ is a bounded sequence}\}$

$B = \{(x_i) \in \mathbb{R}^\omega \mid \{x_i\}_{i=1}^{\infty} \text{ is an unbounded sequence}\}$.

Then $A \cap B = \emptyset$, $A \cup B = \mathbb{R}^\omega$, and we can see that A and B are open as follows: Given a point $(x_i) \in A$, the open set

$$U = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots$$

consists entirely of bounded sequences, so $U \subset A$.

Similarly if (x_i) is unbounded then

$$V = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots$$

is entirely unbounded sequences so $V \subset B$.

Thus $\{A, B\}$ is a separation of $\mathbb{R}^\omega$.

<u>Example</u>: Connectedness of $\mathbb{R}^\omega$ with the product topology.

Let $\overline{\mathbb{R}}^n \subset \mathbb{R}^\omega$ denote the set of all sequences $(x_1, x_2, \dots)$ such that $x_i = 0 \ \forall i > n$. Then $\overline{\mathbb{R}}^n$ is homeomorphic to $\mathbb{R}^n$, and so it is connected. It follows that $\mathbb{R}^\infty = \bigcup_{i=1}^{\infty} \overline{\mathbb{R}^i}$ is connected, since all $\overline{\mathbb{R}}^i$'s have the point $(0, 0, 0, \dots)$ in common. We show that the closure $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$, so that $\mathbb{R}^\omega$ with the product topology is connected as well.

Let $(x_i) \in \mathbb{R}^\omega$. Let $U = \prod U_i$ be a basic open nbhd of (x_i). There exists N s.t. $U_i = \mathbb{R}$ for $i > N$, and thus the point $(x_1, x_2, \dots, x_N, 0, 0, \dots) \in \mathbb{R}^\infty$ belongs to U, since $x_i \in U_i \ \forall i < N$ and $0 \in U_i \ \forall i > N$. Therefore $U \cap \mathbb{R}^\infty \neq \emptyset$, and $\mathbb{R}^\infty$ is dense.