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Last day we saw a construction of

$$S^3 = \{(w,x,y,z) \mid w^2+x^2+y^2+z^2 = 1\}$$

as a union of two tori glued along their boundaries.
Schematically it was:

with a gluing map that reversed the coordinates on the surface of each solid torus. Explicitly, a homeomorphism with S^3 is given by:

The common boundary of the two tori is:

$$\{(w,x,y,z) \in S^3 \mid w^2+x^2 = y^2+z^2 = \tfrac{1}{2}\}$$

and each solid torus is given by

$$\{(w,x,y,z) \in S^3 \mid w^2+x^2 \le \tfrac{1}{2} \text{ and } y^2+z^2 = 1-w^2-x^2\}$$

$$\{(w,x,y,z) \in S^3 \mid y^2+z^2 \le \tfrac{1}{2} \text{ and } w^2+z^2 = 1-y^2-z^2\}.$$

<u>Def of</u> CW complexes here.

A CW complex <u>is finite</u> if it has finitely many cells

A <u>subcomplex</u> of a CW complex is a subspace that is a closed union of cells.

A CW complex of dimension 1 is a graph.

Note that last class we proved a fact we now need:

If D^n is attached to the $(n-1)$-skeleton X^{n-1} by a gluing map, then

$$D^n \xrightarrow{\text{inc}} X^{n-1} \oplus D^n \xrightarrow{\text{quotient}} X^{n-1} \cup_f D^n$$

~~doesn't~~ "preserves" D^n, in the sense that it is an embedding. So we can call the image of D^n in the CW-complex a cell.

Example: The sphere $S^n = \{ (x_1, \dots, x_{n+1}) \mid \Sigma_i^1 x_i^2 = 1 \}$ has two popular decompositions into cells

(i) The 0-skeleton is a single vertex v, and there are no k-cells for $0 < k < n$. There is a single n-cell D^n with gluing $f : \partial D^n \longrightarrow v$ a constant map, an no higher dimensional cells.

Eg. S^2:

(ii) Alternatively, S^n can be decomposed into two n-cells and a copy of S^{n-1}, which we think of as the 'equator'. Specifically, the two cells are

$$e_+^n = \{ (x_0, \dots, x_n) \in S^n \mid x_n \geq 0 \}$$

and

$$e_-^n = \{ (x_0, \dots, x_n) \in S^n \mid x_n \leq 0 \}$$

with $S^{n-1} = \{ (x_0, \dots, x_n) \in S^n \mid x_n = 0 \}$.

Then we can decompose S^{n-1} into two cells in the same way. In the end we have cells $\{e^i_\pm\}_{i=0}^{n}$

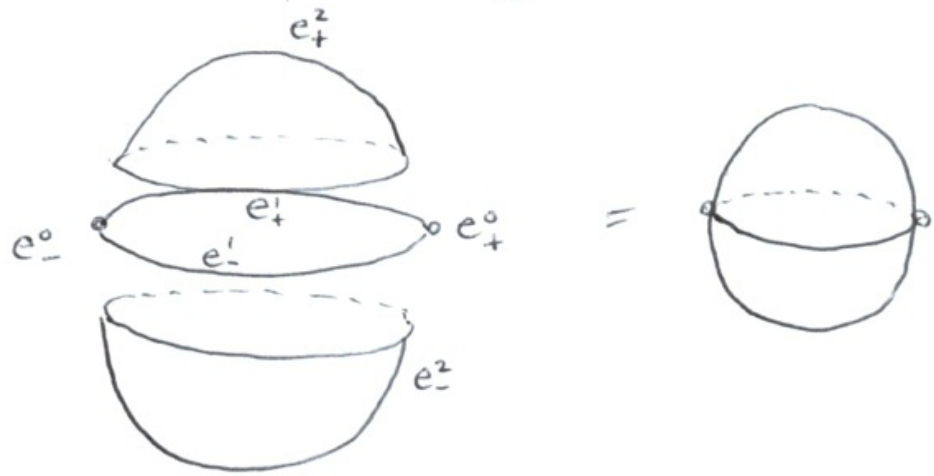
with $$e^i_+ = \{(x_0,\dots,x_i,0,\dots,0)\in S^n \mid x_i \geq 0\}$$

$$e^i_- = \{(x_0,\dots,x_i,0,\dots,0)\in S^n \mid x_i \leq 0\},$$ with implicit gluing maps.

E.g for S^2 we have:

0-skeleton 1-skeleton

2-skeleton:



Examples: The earlier constructions of the torus and the projective plane can also be realized as CW-complexes.

Chapter 5: Product Spaces.

The product of two spaces X and Y is $X\times Y$ with the initial topology induced

By the projection maps $X \longleftarrow X \times Y \longrightarrow Y$.
In other words, the topology has basis $\{U \times V \mid U \subset X, V \subset Y\}$, called the standard basis.

<u>Proposition</u>: If X is an n-manifold and Y is an m-manifold, then $X \times Y$ is an $(m+n)$-manifold.

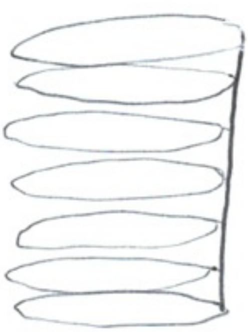
<u>Proof</u>: Let $(x,y) \in X \times Y$. Then let U be an open neighbourhood of $x \in U \subset X$ s.t. $U \cong \mathbb{R}^n$, V an open nbhd of $y \in Y$ s.t. $V \cong \mathbb{R}^m$. Then $U \times V$ is an open nbhd of $(x,y) \in X \times Y$ and $U \times V \cong \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{(n+m)}$.

<u>Example</u>: The torus $T^2 = S^1 \times S^1$ is a 2-manifold since S^1 is a 1-manifold.

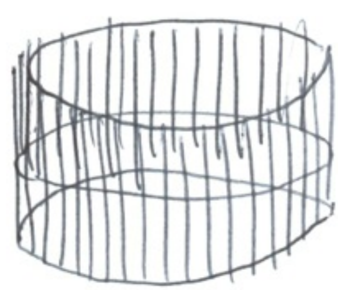
<u>Proposition</u>: $X \times Y \cong Y \times X$ via the homeomorphism
$$f: X \times Y \longrightarrow Y \times X, \quad f(x,y) = (y,x).$$

<u>Proof</u>: Clearly f is bijective. Also for every basic element $V \times U \subset Y \times X$, $f^{-1}(V \times U) = U \times V$ is a basic element of $X \times Y$.
Therefore f is continuous. By symmetry, f is open.

<u>Example</u>: The space $[0,1] \times S^1$ is



vs. $S^1 \times [0,1]$ which is



Remark: Introduced this way, there is some issue as to whether or not $(X \times Y) \times Z \simeq X \times (Y \times Z)$. Thankfully these spaces are homeomorphic (the issue is that our basis would depend on the order of the parentheses). So from here on we disregard parentheses.

Proposition: Let $\{X_i\}_{i=1}^n$ be topological spaces.

(i) If the X_i are all first countable, so is $\prod_{i=1}^n X_i$

(ii) If the X_i are all second countable, so is $\prod_{i=1}^n X_i$

(iii) If the X_i are all separable, so is $\prod_{i=1}^n X_i$

(iv) If the X_i are all metrizable, then so is $\prod_{i=1}^n X_i$

Proof:

(i) Given $x = (x_1, \dots, x_n) \in \prod_{i=1}^n X_i$, each $x_i \in X_i$ has a local basis $\mathcal{B}_i$.

Set $\mathcal{B}_x = \{ \prod_{i=1}^n B_i \mid B_i \in \mathcal{B}_i \}$. Then $\mathcal{B}_x$ is countable, and it is also a local basis at x: Let $U \subset \prod_{i=1}^n X_i$ be an open nbhd of x. ~~Since~~ Then there is a standard basis element $\prod_{i=1}^n U_i$ with

$x \in \prod_{i=1}^n U_i \subset U$. Since each $\mathcal{B}_i$ is a local basis, for each i $\exists B_i \subset U_i$ with $x_i \in B_i \subset U_i$. Then

$x \in \prod_{i=1}^n B_i \subset \prod_{i=1}^n U_i \subset U$ shows that $\mathcal{B}_x$ is a local basis at x.

(iv) Suppose each X_i has metric d_i, which generates the topology on X_i. On $\prod_{i=1}^{n} X_i$ there is a metric d defined as follows: If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ then $d(x,y) = \sqrt{d_1(x_1,y_1)^2 + \dots + d_n(x_n,y_n)^2}$.

Then we must check that the topology generated by this metric is the same as the product topology on $\prod_{i=1}^{n} X_i$ (Exercise).

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Infinite products.

Let $\{X_i\}_{i\in I}$ be a collection of topological spaces. Then elements of the product $\prod_{i\in I} X_i$ are formally maps $f: I \longrightarrow \bigcup_{i\in I} X_i$ where the union is disjoint.

However, instead of writing $f \in \prod_{i\in I} X_i$, we think of elements of $\prod_{i\in I} X_i$ as 'sequences' $(x_i)_{i\in I}$, where $f(i) = x_i$ are the coordinates of (x_i).

As we already saw, the product topology on $\prod_{i\in I} X_i$ is the initial topology with respect to the maps $p_i : \prod_{i\in I} X_i \longrightarrow X_i$. As such the definition of the initial topology gives us a subbasis

$$S = \{U \subset \prod_{i\in I} X_i \mid U = p_i^{-1}(V) \text{ for some } i, V \subset X_i \text{ open}\}.$$

Every such U is of the form:

$$\left(\prod_{i\in I\setminus\{j\}} X_i\right) \times V,$$ ie. it's X_i in all coordinates except one, where it's V.

As such the basis of the product topology is:

Proposition: The product topology over X_i is generated by all sets of the form $\prod_{i\in I} U_i$ where for every $i \in I$, U_i is an open subset of X_i, and $U_i = X_i$ for all but finitely many i.

There is another 'natural' seeming topology on $\prod_{i\in I} X_i$, namely the topology whose basis is

$$\mathcal{B} = \{\prod_{i\in I} U_i \mid U_i \subset X_i \text{ open}\}.$$

The resulting topology is called the box <u>topology</u>.
Note that the box topology and the product topology coincide if the index set is finite, but different in essential ways if I is infinite.

Recall:

<u>Proposition</u>: (The infinite version of an assignment question)
Suppose $f: Y \longrightarrow \prod_{i\in I} X_i$ is a map. For each $j\in I$, let $f_j: Y \longrightarrow X_j$ denote the map $p_j f: Y \to \prod_{i\in I} X_i \to X_j$, called the j^{th} component of f.
Then f is continuous if and only if f_j is continuous $\forall j \in I$.

On the other hand, in the box topology we have:

<u>Example</u>: Let $f: \mathbb{R} \longrightarrow \prod_{i=1}^{\infty} \mathbb{R}$ be the map whose coordinate functions f_i are $f_i(x) = x$ for all i.
Equip $\prod_{i=1}^{\infty} \mathbb{R}$ with the box topology, and consider the set $\prod_{i=1}^{\infty} (\frac{-1}{n}, \frac{1}{n}) \subset \prod_{i=1}^{\infty} \mathbb{R}$. Since we are working in the box topology, this set is open. However

$$f^{-1}\left(\prod_{i=1}^{\infty}\left(\tfrac{-1}{n}, \tfrac{1}{n}\right)\right) = \bigcap_{i=1}^{\infty}\left(\tfrac{-1}{n}, \tfrac{1}{n}\right) = \{0\},$$ so it is not open.

Therefore f is a function with continuous components, but f itself is not continuous.

We also have

<u>Proposition</u>: Let $\{X_i\}_{i=1}^{\infty}$ be a countable collection of spaces. Then $\prod_{i=1}^{\infty} X_i$ is first countable iff X_i is first countable $\forall i$.

<u>Proof</u>: ($\Rightarrow$) Each of the projection maps $p_i: \prod_{i=1}^{\infty} X_i \longrightarrow X_i$ is a continuous, open map, and we have the following lemma.

<u>Lemma</u>: If $f: X \longrightarrow Y$ is continuous and open and surjective, then X first countable $\Longrightarrow$ Y first countable.

Thus, by applying the lemma to all projection maps p_i we find that each X_i is first countable.

($\Leftarrow$). Let $(x_i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} X_i$ be given. Let $\mathcal{B}_i$ be a countable local basis of x_i, $i \in \mathbb{N}$, and let $\mathcal{F}$ denote the set of all finite subsets of $\mathbb{N}$, $\mathcal{F}$ is countable. For each $F \subset \mathcal{F}$, set

$$\mathcal{B}_F = \left\{ \prod_{i=1}^{\infty} U_i \;\middle|\; U_i = X_i \text{ for } i \notin F \text{ and } U_i \in \mathcal{B}_i \text{ for } i \in F \right\}$$

Since each $\mathcal{B}_i$ is countable and F is finite, $\mathcal{B}_F$ is countable, and each element of $\mathcal{B}_F$ is an open nbhd of $(x_i) \in \prod_{i=1}^{\infty} X_i$. Set

$$\mathcal{B}_x = \bigcup_{F \in \mathcal{F}} \mathcal{B}_F .$$

Then because each $\mathcal{B}_F$ is countable and $\mathcal{F}$ is countable, $\mathcal{B}_x$ is also countable.

From here it is a notation-heavy technical check to verify that $\mathcal{B}_x$ is a local basis at x.

(omitted)

On the other hand, we have:

<u>Proposition</u>: $X = \prod_{i=1}^{\infty} \mathbb{R}$, equipped with the box topology, is not first countable.

<u>Proof</u>: Let $x=(x_i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty}\mathbb{R}$ be given, and suppose $\mathcal{B}_x = \{B_j\}_{j=1}^{\infty}$ is a countable local basis at $(x_i)=x$.

By replacing each B_j with an open set $\prod_{i=1}^{\infty} U_{j,i} \subset B_j$ if necessary, we can assume that $B_j = \prod_{i=1}^{\infty} U_{j,i}$ for all j; with $U_{j,i}$ non-empty and open in $\mathbb{R}$ for all j.

(Because sets of the form $\prod U_i$ are a basis for the box topology).

Let V_i be a proper, open subset of $U_{i,i}$ for $i=1,2,\dots$ containing x_i.

Then $\prod_{i=1}^{\infty} V_i$ is an open nbhd of $(x_i)_{i\in I}$.

However, there is no $B_j \in \mathcal{B}_x$ with $x \in B_j \subset \prod_{i=1}^{\infty} V_i$, by our construction of V_i.

(To see that $B_j \not\subset \prod_{i=1}^{\infty} V_i$, we need only compare $B_j = \prod_{i=1}^{\infty} U_{j,i}$ and $\prod_{i=1}^{\infty} V_i$ in the j-th coordinate).

<u>Remark</u>: The previous two propositions hold if first countable is replaced second countable or metrizable.

This is not to say that product spaces are wonderful.

<u>Example</u>: We construct the Cantor set $C \subset [0,1]$ by using the middle-thirds construction.

Set $C_1 = [0,1]$, $C_2 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$, and in general

$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$. (remove the middle thirds of previous intervals).

Set $C = \bigcap_{i=1}^{\infty} C_n \subset [0,1]$, with the subspace topology.

Which numbers in $[0,1]$ are in C?

Write all numbers in $[0,1]$ in their base 3 decimal expansion, $0.x_1x_2x_3x_4\ldots$ where $x_i \in \{0,1,2\}$ $\forall i$.

Removing the first middle third deletes all numbers with $x_1 = 1$.

Removing the middle thirds from the remaining bits deletes all numbers with $x_2 = 1$, etc.

In general,

$$C = \bigcap_{i=1}^{\infty} C_n$$

consists of all numbers in $[0,1]$

whose base 3 expansion contains only 0's and 2's. Define a map $f: C \longrightarrow \prod_{i=1}^{\infty} \{0,2\}$ by

$$f(0.x_1 x_2 x_3 x_4 \ldots) = (x_i)_{i \in \mathbb{N}},$$

where $\{0,2\}$ is the two-point discrete space. Then incredibly, f is a homeomorphism.