MATH 3240 Topology 1 Feb 4.

Lecture 9

The mapping $h: A \cup B \to Y$ in the gluing lemma is called the union of the maps f and g.

Definition: Let $\{(X_i, \tau_i)\}_{i \in I}$ be a collection of pairwise disjoint spaces. Give $\bigcup_{i \in I} X_i$ the topology τ with basis

$$\mathcal{B} = \{U \subset \bigcup_{i \in I} X_i \mid U \in \tau_i \text{ for some } i\}.$$

Then with this topology, $\bigcup_{i \in I} X_i$ is denoted $\bigoplus_{i \in I} X_i$, and is called the ~~discrete sum~~ topological sum, or coproduct.

Proposition: $U \subset \bigoplus_{i \in I} X_i$ is open if and only if $U \cap X_i \in \tau_i$ for all i.

Proof: Exercise (might be on assignment).

Remark: Note that we can define the topology on $\bigoplus_{i \in I} X_i$ a second way: consider the inclusion maps $f_i : X_i \longrightarrow \bigoplus_{i \in I} X_i = \bigcup_{i \in I} X_i$. The finest topology on $\bigcup_{i \in I} X_i$ making f_i continuous for all i (i.e. the final topology induced by $\{f_i\}_{i \in I}$) is

$\mathcal{T} = \{ U \subset \bigcup_{i \in I} X_i \mid f_i^{-1}(U) \text{ is open in } X_i \text{ for all } i \}$.

But since f_i is an inclusion map, $f_i^{-1}(U) = U \cap X_i$ so

$\mathcal{T} = \{ U \subset \bigcup_{i \in I} X_i \mid U \cap X_i \text{ is open in } X_i \text{ for all } i \}$,

which is exactly the description of $\mathcal{T}$ in the last proposition. ! This description doesn't require disjoint X_i !

Def: In more generality, if $\mathcal{A} = \{X_i\}_{i \in I}$ is a collection of non-disjoint sets then

$$\mathcal{T}(\mathcal{A}) = \{ U \subset \bigcup_{i \in I} X_i \mid U \cap X_i \text{ is open in } X_i \text{ for every } i \in I \}$$

is called the weak topology over $\bigcup_{i \in I} X_i$ determined by $\mathcal{A}$. ~~(Note~~.

We can use these constructions, together with the gluing lemma, to glue together disjoint spaces (not just subsets of the same space).

Def: Let X and Y be disjoint spaces, and $A \subset X$ nonempty. Let $f: A \to Y$ be any map, and write $B = f(A)$.
Define an equivalence relation on $X \cup Y$ by

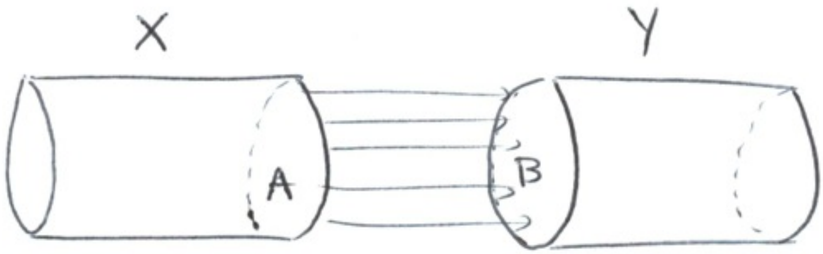
$a \sim b$ if $f(a) = b$, and $x \sim x$ for all $x \in X \cup Y$.

So the equivalence classes are

$\{ \{y\} \cup f^{-1}(\{y\}) \}_{y \in B}$, and singletons.

Then $X \oplus Y/\sim$ is called the space obtained by gluing X and Y along f, or gluing X and Y along A and B via f. We write $X \cup_f Y$ in place of $X \oplus Y/_\sim$.

Picture:

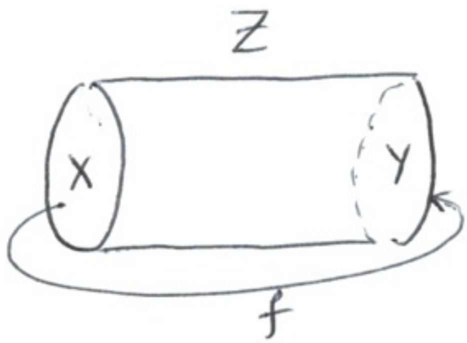


Then $X \cup_f Y =$ , with the topology you expect.

In the special case where $X, Y \subset Z$ are disjoint subspaces and $A = X$, then there is an equivalence relation on Z, similar to before: Since $f : X \to Y$ we define the equivalence classes to be

$$\{\{y\} \cup f^{-1}(y)\}_{y \in f(X)}, \text{ and singletons.}$$

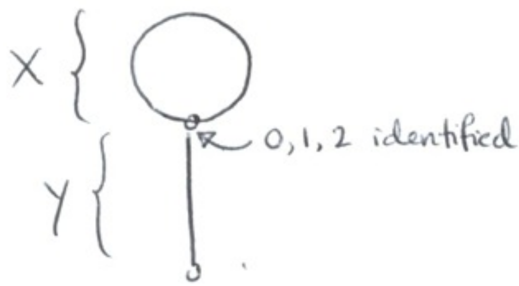
Then $Z/\sim$ is obtained from Z by gluing A and B via f.



, then $Z/_\sim$ is

So $X \cup_f Y$ comes from pasting disjoint spaces, Z_f is gluing a space to itself.

<u>Example</u>: Consider $X = [0,1]$ and $Y = [2,3]$.
Set $A = \{0,1\}$ and $f : A \longrightarrow Y$ be $f(0) = f(1) = 2$.
Then $X \cup_f Y$ is the space

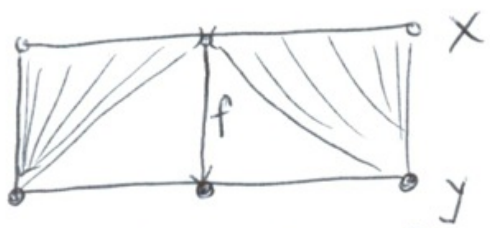


We could also set $A = X$ and define $f : A \longrightarrow Y$ by

$$f(x) = \begin{cases} 2 & \text{if } x < \frac{1}{2} \\ 5/2 & \text{if } x = \frac{1}{2} \\ 3 & \text{if } x > \frac{1}{2} \end{cases}$$

(Recall: f need not be continuous in this example).

Then $X \cup_f Y$ is a horrible space:



In particular, note that the image of Y in the quotient $X \oplus Y/\sim$ is bad: Let $q : X \oplus Y \longrightarrow X \oplus Y/\sim$ be the quotient.
If $U \subset X \oplus Y/\sim$ contains the image $q(5/2)$, then $q^{-1}(U)$ is an open union of equivalence classes. But then $q^{-1}(U)$ must contain an open nbhd of $f^{-1}(\frac{5}{2}) = \frac{1}{2}$,

hence $q^{-1}(U)$ must contain the equivalence classes $(\frac{1}{2}, 1] \cup \{3\}$ and $[0, \frac{1}{2}) \cup \{2\}$. In particular, every open neighbourhood of $q(\frac{5}{2})$ will also contain $q(2)$ and $q(3)$. Thus the image of Y in $X \oplus Y/\sim$ is bad.

<u>Lesson learned</u>: Neither of these:

$$X \xrightarrow{\text{inclusion}} X \oplus Y \longrightarrow X \oplus Y/\sim$$

$$Y \xrightarrow{\text{inclusion}} X \oplus Y \longrightarrow X \oplus Y/\sim$$

is an embedding (homeomorphism onto its image).
In the first case, we had X folded on itself so the map was not injective.
In the second case, the topology on Y was terrible.

In general, though, the structure of Y is preserved:

<u>Proposition</u>: Suppose X and Y are topological paces and $A \subset X$ closed. If $f: A \to Y$ is continuous, then

$$Y \xrightarrow{\text{inclusion}} X \oplus Y \longrightarrow X \oplus Y/\sim = X \cup_f Y$$

is an embedding.

<u>Proof</u>: Give the composition above a name, call it $g: Y \to X \cup_f Y$. The formula is $g(y) = [y]$.

First, g is injective: If $g(y_1)=g(y_2)$ then $[y_1]=[y_2]$. The equivalence class of $y \in Y$ is $\{y\} \cup \underbrace{f^{-1}(y)}_{\text{subset of } X}$, so if $[y_1]=[y_2]$ then

y_2 is not in here. y_1 is not in here

$$\{y_1\} \cup f^{-1}(y_1) = \{y_2\} \cup f^{-1}(y_2)$$

so $y_1 = y_2$.

Also, g is continuous, since it is the composition of

$Y \xrightarrow{\text{inc.}} X \oplus Y$ (by definition of the topology on $X \oplus Y$, this is continuous)

and $X \oplus Y \longrightarrow X \oplus Y/\sim$, also continuous by definition of the topology on $(X \oplus Y)/\sim$.

We show g is an embedding by checking it is a closed map $g : Y \to g(Y)$.

Suppose $B \subset Y$ is closed. Then

$g(B) = \{[y] \in X \cup_f Y \mid y \in B\} = \{\{y\} \cup f^{-1}(y) \mid y \in B\}$. It follows that $g(B)$ is closed if and only if $\bigcup_{y \in B} (\{y\} \cup f^{-1}(y))$ is closed in $X \oplus Y$. But observe that

$$\bigcup_{y \in B} \{y\} \cup f^{-1}(y) = \bigcup_{y \in B} \{y\} \cup \bigcup_{y \in B} f^{-1}(y) = B \cup f^{-1}(B),$$ ~~which~~

and $f^{-1}(B)$ is closed in A since f is continuous.

Thus there is a closed $C \subset X$ such that
$f^{-1}(B) = C \cap A$. Therefore

$B \cup f^{-1}(B) = B \cup (C \cap A)$, and considered as a subset
of $X \oplus Y$ this is closed since:

$(B \cup (C \cap A)) \cap Y = B$ is closed, and

$(B \cup (C \cap A)) \cap X = C \cap A$ is closed since A is closed.

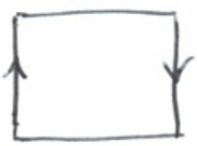
Thus $g(B)$ is closed in $\underline{\underline{X \cup_f Y}}$.

MATH 3240 Topology 1 (Feb 6).

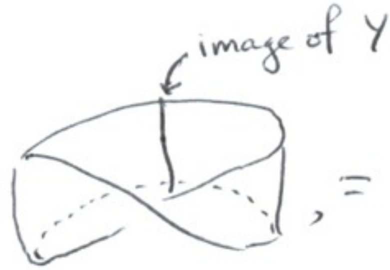
Lecture 10

The moral of the last proposition is:
If we want to consider spaces $X \cup_f Y$ with reasonably nice topologies, we restrict to $A \subset X$ closed and $f: A \to Y$ continuous.

<u>Example</u>: Let $Z = [0,1] \times [0,1]$. Set $X = \{0\} \times [0,1]$, $Y = \{1\} \times [0,1]$, and define $f: X \to Y$ by $f(0,t) = (1, 1-t)$. Schematically,



Then Z_f is a Möbius band



$, = M,$

and the image of Y in Z_f appears as an interval which cuts M into a single untwisted strip.

<u>Example</u>. Let $X = S^2 \setminus \text{disk}$, $A = \partial X$. Specifically

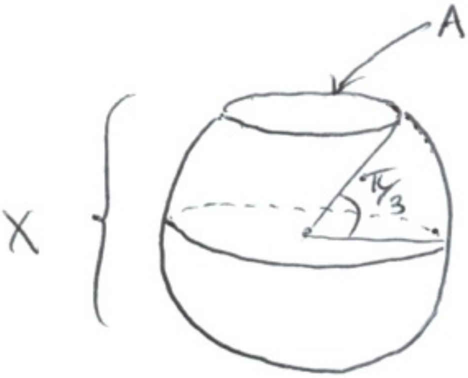
$X = \{ (x,y,z) \mid x^2+y^2+z^2 = 1,\ z \leq \sqrt{3}/2 \}$ and

$A = X \cap \{ (x,y,z) \mid z = \sqrt{3}/2 \}$.

Let $Y = Z_f = [0,1] \times [0,1] / \sim$ from the previous example.

In spherical coordinates, $A = \{ (r, \theta, \phi) \mid r = 1,\ \phi = \pi/3,\ \theta \in [0, 2\pi] \}$.

Picture:



, $Y =$ 

Define $f: A \longrightarrow Y$ by $f(r,\theta,\phi) = \begin{cases} \left[\left(\frac{\theta}{\pi}, 0\right)\right] & \text{if } 0 \le \theta \le \pi \\ \left[\left(\frac{\theta-\pi}{\pi}, 1\right)\right] & \text{if } \pi \le \theta \le 2\pi \end{cases}$

In pictures:

The resulting space is a sphere with an attached Möbius band, sewn along a circle S^1.

<u>Fact</u>: $X \cup_f Y$ is homeomorphic to the real projective plane. The sewing of Y is sometimes called a "cross cap".

Some of the types of spaces we construct in this way are (§4.4). <u>Manifolds</u> and CW-complexes.

Definition: A space X is a manifold (an n-manifold) if the following conditions are satisfied:

(i) X is Hausdorff.

(ii) X is second countable

(iii) For every $x \in X$ there's an open nbhd U of x such that U is homeomorphic to $B^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 < 1\}$.

Here, n is the dimension of the manifold X.

Note that the dimension is well-defined if and only if $\mathbb{R}^n \simeq \mathbb{R}^m \Rightarrow n = m$.

This follows from

Theorem (Invariance of Domain, Brouwer 1912).

If $U \subset \mathbb{R}$ is open and $f: U \longrightarrow \mathbb{R}^n$ is an embedding, then $f(U)$ is open. in $\mathbb{R}^n$.

(Proof is well beyond this course, uses algebraic topology).

Corollary: $\mathbb{R}^n \simeq \mathbb{R}^m$ iff $n = m$.

Proof: Let $g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a homeomorphism and suppose WLOG that $n < m$.

Let $\emptyset \neq U \subset \mathbb{R}^m$ be an open set, then $g^{-1}(U)$ is open, nonempty.

Fix numbers $a_{n+1}, a_{n+2}, \dots, a_m \in \mathbb{R}$ and define

$c: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $c(x_1, \dots, x_n) = (x_1, \dots, x_n, a_{n+1}, \dots, a_m)$.

Then $c \circ g^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is an embedding, and the invariance of domain theorem implies that $c \circ g^{-1}(U)$ is open. However this is not possible, the image of $c : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is $\mathbb{R}^n \times \{a_{n+1}\} \times \dots \times \{a_m\} \subseteq \mathbb{R}^m$, which has empty interior. Contradiction.

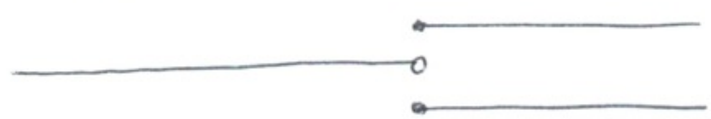
<u>Eg.</u> If $m = 2$ and $n = 1$ then the image of c would be $\mathbb{R} \times \{a_2\}$, a horizontal line

<u>Remarks</u>: Some texts allow manifolds to be non-Hausdorff. This does not immediately follow from the most crucial property, which is (iii).

Eg. Consider $\mathbb{R}\times\{0\} \oplus \mathbb{R}\times\{1\} \big/ \sim \;= X$

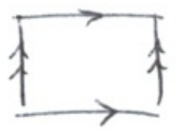
where $(x, 0) \sim (y, 1)$ whenever $x = y$ and $x < 0$.

So we have



a "double zero". The two zeros $(0,0)$ and $(0,1)$ are not Hausdorff separated, but every point in X has a neighbourhood homeomorphic to $(-\varepsilon, \varepsilon)$.

Example: The real projective plane is a 2-manifold, so is the torus



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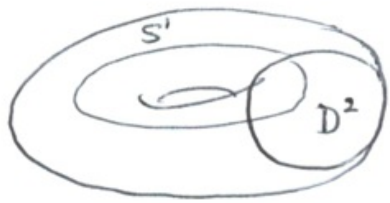
Example: Consider the solid torus $D^2 \times S^1$



Take two disjoint solid tori $X = D^2 \times S^1 \times \{0\}$ and $Y = D^2 \times S^1 \times \{1\}$. Let $A \subset X$ be the surface of X, specifically $A = \partial D^2 \times S^1 \times \{0\}$. Note that $\partial D^2 \simeq S^1$, so we can define a map $f : A \longrightarrow Y$ by $f(x,y,0) = (y,x,1)$. This glues the two surfaces of X and Y by "switching coordinates".

Picture:

The resulting space cannot be drawn in 3d. It is the 3-sphere (homeomorphic to it, at least)

$$S^3 = \{(x_1, x_2, x_3, x_4) \subseteq \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

Definition: A CW-complex is constructed by gluing cells. An n-cell is

$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$

i.e. a 0-cell is a point, a 1-cell is $[-1,1]$, a 2-cell is the unit disk, etc.

A CW-complex X is constructed as follows:
Begin with a discrete space, the 0-skeleton of X, written X^0.
Attach 1-cells to X^0 via maps
$f: \partial[-1,1] = \{-1,1\} \longrightarrow X^0$. Such maps are always continuous since $\{-1,1\}$ is a discrete space. The resulting space is the 1-skeleton X^1.

E.g.

Assuming we have constructed the n-skeleton X^n, the $(n+1)$-skeleton is obtained by attaching $(n+1)$-cells D^{n+1} via continuous $f: \partial(D^{n+1}) = S^n \longrightarrow X^n$.

If this procedure terminates, the dimension of X is the dimension of the highest dimensional cell.

Equivalently:

Def: A CW complex is a space X and a collection of disjoint open cells $\{e_\alpha\}_{\alpha \in A}$ whose union is X, satisfying

(1) X is Hausdorff

(2) For each e_α an m-cell, there exists a continuous map $f_\alpha : D^m \to X$ mapping D^m homeomorphically onto e_α and carrying ∂D^m into a finite union of open cells of dimension less than m.

(3) A set $F \subset X$ is closed in X if $F \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for each $\alpha \in A$.