Topology 1 January 28. (Assignments!)
Lecture 7

Last day, ended with:

Proposition: If $f: X \to Y$ is a homeomorphism and $A \subset X$, then
(i) $f(\mathrm{int}(A)) = \mathrm{int}(f(A))$
(ii) $f(\bar{A}) = \overline{f(A)}$
(iii) $f(\partial A) = \partial(f(A))$
(iv) $f(A') = (f(A))'$.

Proof: We omit. It is mostly definition-checking.

Definition: Let X be a set (no topology on it), and Y a space, and $f: X \to Y$ a map. The weakest topology on X making f continuous is called the topology induced by f. (initial topology)
In general, if $\{f_i : X \to Y_i\}_{i \in I}$ is a family of maps, the topology induced by the f_i on X is the weakest (coarsest) topology making them all continuous.

Proposition: Given $f_i : X \to Y_i$ as above, the topology on X induced by $\{f_i\}_{i \in I}$ has subbasis

$$S = \{ f_i^{-1}(U) \mid U \text{ is open in } Y_i,\ i \in I \}.$$

Proof: We name all the topologies involved. Let τ_i be the topology on Y_i, and let τ be the topology with subbasis S. We must show that all f_i are continuous, and τ is the coarsest topology that makes this happen.

First, each f_i is continuous because $U \in \tau_i \Rightarrow f^{-1}(U) \in \tau$ by definition.

Suppose τ' is another topology on X making f_i continuous for all i. Then by continuity, $f_i^{-1}(U) \in \tau'$ for all $U \in \tau_i$, $\forall i$. But then $S \subset \tau'$ and so $\tau \subset \tau'$, since τ is (by definition of subbasis/basis) the coarsest topology containing S.

Example:

Let $f_i : X \to Y_i$ be the projection maps $p_i : \prod_{i \in I} Y_i \longrightarrow Y_i$. The product topology on $Y = \prod_{i \in I} Y_i$ is the topology induced by the p_i.

Therefore a subbasis for the product topology is

$S = \{ p_i^{-1}(U) \mid U \text{ is open in } Y_i,\ i \in I \}$.

The set $p_i^{-1}(U)$ looks like (here if $I = \mathbb{Z}$)

$$\times Y_1 \times Y_2 \times \dots \times Y_{i-1} \times U_i \times Y_{i+1} \times \dots \subset \prod_{i \in \mathbb{Z}} Y_i$$

U_i in the i^{th} factor.

We investigate the product topology more in Ch 5.

We can also induce a topology in a dual sense:

<u>Definition</u>: Given a family of topological spaces Y_i with maps $f_i : Y_i \longrightarrow X$ into a set X, the finest topology τ on X making $f_i : Y_i \longrightarrow X$ continuous for all i is called the <u>final topology</u> (again, topology induced by f_i).

$\tau = \{ U \subset X \mid f_i^{-1}(U) \text{ is open for all } i \}$.

<u>Examples</u> of initial and final topologies:

<u>§4.1</u> Subspaces. and <u>§4.2</u> Quotients.

Let X be a topological space and $A \subset X$ a subset.

<u>Proposition</u>: The set $\tau_A = \{ A \cap U \mid U \text{ is open in } X \}$ is a topology on A.

<u>Proof</u>: Let $i : A \hookrightarrow X$ be the inclusion map, $i(a) = a$ $\forall a \in A$. Then τ_A is the initial topology on A induced by i, hence it is a topology.

<u>Definition</u>: The set $\tau_A = \{ A \cap U \mid U \text{ open in } X \}$ is the <u>subspace topology</u> on A.

<u>Proposition</u>: A subset U of $A \subset X$ is closed in A (with the subspace topology) iff $\exists\ V$ closed in X such that $U = V \cap A$.

Proof: ($\Rightarrow$). Suppose U is closed. Then U^c is open, so by definition $\exists\, W \subset X$ open such that $W \cap A = U^c$. But then W^c is closed and $W^c \cap A = U$, done.

($\Leftarrow$) If W is closed (in X) with $U = A \cap W$, then $U^c = A \cap W^c$ is open (by definition), hence U is closed.

Convention: From now one we will always assume that subsets of a space carry the subspace topology, unless othewise stated.

Terminology:

- A map $f: X \to Y$ is an embedding if f is a homeomorphism onto its image $f(X)$. ($f(X)$ with subspace topology).
- Given a map $f: X \to Y$ (continuous) and a subspace $A \subset X$, the map $f|_A : A \to Y$ ~~be~~ with $f|_A(a) = f(a)$ is the restriction map. It is continuous with respect to the subspace topology on A.

Definition: A property of a space X is hereditary if, whenever X has that property then so does every subspace of X.

Example:

(a) Second countability is hereditary.

Proof: Suppose X is second countable and $A \subset X$. If $\mathcal{B}$ is a countable basis of X, then

$$\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$$ is a countable basis for A.

To see this, suppose that $U \subset A$ is open. Then there exists $V \subset X$ open s.t. $U = A \cap V$. Write $V = \bigcup_{i \in I} B_i$ for some $B_i \in B$. Then

$$U = A \cap \left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} (A \cap B_i),$$ hence U is a union of elements of $\mathcal{B}_A$. Thus $\mathcal{B}_A$ is a countable basis.

(b) First countability is hereditary.

Proof. Let $A \subset X$ and let $a \in A$ be given. Since X is first countable and $a \in X$, there exists a countable local basis $\mathcal{B}_a$ at a. Then

$$\mathcal{B}_a' = \{A \cap B \mid B \in \mathcal{B}_a\}$$ is a countable local basis for $a \in A$. To see this, suppose $U \subset A$ is open and $a \in U$. Then $\exists\, V \subset X$ open such that $U = A \cap V$. Thus $\exists\, B \in \mathcal{B}_a$ s.t. $a \in B \subset V$.

But then $a \in \underset{\mathcal{B}_a'}{\underset{\cap}{B \cap A}} \subset V \cap A \subset U$. Thus $\mathcal{B}_a'$ is a countable local basis.

Example: Separability is not hereditary.

Proof: Recall the Sorgenfrey line is a copy of $\mathbb{R}$ with the topology generated by the basis

$$\mathcal{B} = \{[a,b) \mid \underset{a<b}{a,b \in \mathbb{R}}\}.$$

The Sorgenfrey plane is $\mathbb{R}^2$ together with the product topology, whose basis is

$$\{[a,b) \times [c,d) \mid \underset{a<b}{(a,c) \in \mathbb{R}\times\mathbb{R}}, \underset{c<d}{(b,d) \in \mathbb{R}\times\mathbb{R}}\}.$$

So open basis elements look like:

The Sorgenfrey plane is separable, because $\mathbb{Q}^2$ is countable and dense.

Set $A = \{(x,-x) \mid x \in \mathbb{R}\}$. Then A is not separable, because it is uncountable and the subspace topology is discrete: To see this, we first show discrete:

Given any $(x,-x) \in A$, $U = [x, x+1) \times [-x, -x+1)$ is open and $U \cap A = (x,-x)$.

Thus every $\{(x,-x)\} \subset A$ is open, so every subset is open.

§4.2 Quoteints

Let X be a topological space and $\sim$ an equivalence relation on X. Denote the equivalence class of $x \in X$ by $[x]$, and let $X/\sim$ denote the set of equivalence classes.

The quotient topology on $X/\sim$ is the final topology induced by the map
$$q : X \longrightarrow X/\sim, \quad x \longmapsto [x].$$

I.e.

Proposition: A subset $\mathcal{U} \subset X/\sim$ is open iff $\underbrace{\bigcup_{[x]\in\mathcal{U}} [x]}$ is open in X.

Here, $[x]$ is a subset of X, and $\mathcal{U} \subset X/\sim$ is a collection of such subsets; i.

Proof: By definition of the final topology and the map q.

Definition: The topology induced by the quotient map $x \longmapsto [x]$ is called the quotient topology, and q a quotient map.

Topology 1 January 30

Lecture 8

§4.2 Quotients

Recall X is a topological space, $\sim$ an equivalence relation and $X/\sim$ the set of equivalence classes $[x] \subset X$.

Then the quotient topology on $X/\sim$ is the final topology induced by the map $q: X \longrightarrow X/\sim$, $x \longmapsto [x]$.

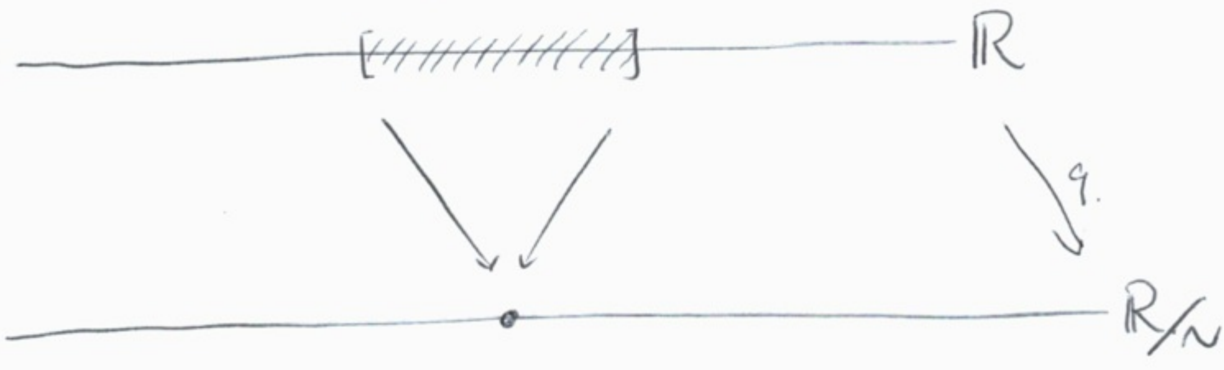
Prop: $U \subset X/\sim$ is open iff $\bigcup_{[x]\in U} [x]$ is open in X, similarly $F \subset X/\sim$ is closed (in $X/\sim$) iff $\bigcup_{[x]\in F} [x]$ is closed in X.

Proof: By definition of the final topology.

Also: $\tau_Y = \{U \subseteq \underset{X/\sim}{\underset{\parallel}{Y}} \mid q^{-1}(U) \in \tau_X\}$

Example:

Consider $\mathbb{R}$ together with $\sim$ defined by $x \sim y$ if either $x=y$ or $x,y \in [a,b]$. Then $\mathbb{R}/\sim$ is $\mathbb{R}$, with $[a,b]$ collapsed to a point:

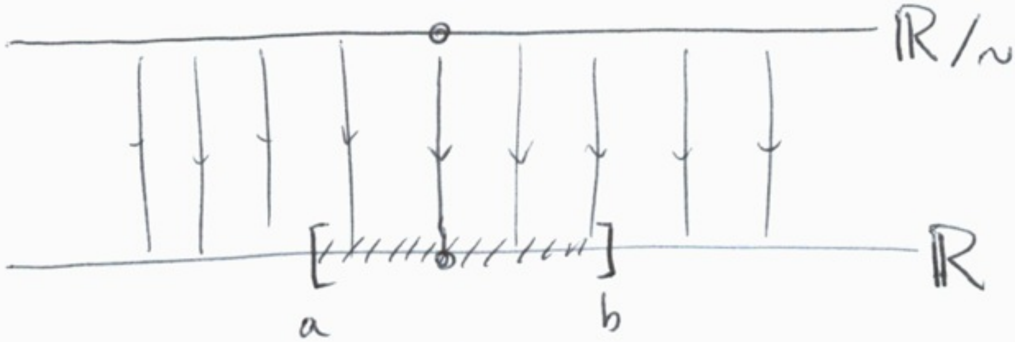


The quotient is actually homeomorphic to $\mathbb{R}$.

Define a map $h : \mathbb{R}/\sim \longrightarrow \mathbb{R}$ by

$$h([x]) = \begin{cases} x - \left(\frac{b-a}{2}\right) & \text{if } x > b \\ \frac{b+a}{2} & \text{if } a \le x \le b \text{ (one equivalence class)}. \\ x + \left(\frac{b-a}{2}\right) & \text{if } x < a. \end{cases}$$

This is



Then h is clearly well-defined on equivalence classes, and is bijective. Moreover h is continuous: the open basic set $(c,d) \subset \mathbb{R}$ has inverse image $h^{-1}(c,d) = (h^{-1}(c), h^{-1}(d))^{\text{(open)}}$, here we are using interval notation in the set $X/\sim$. This is ok since it obviously inherits an ordering.

<u>Example</u>: Consider $\mathbb{R}$ with $\sim$ defined by $x \sim y$ if $x = y$ or $x, y \in (a,b)$.

Then the ~~points~~ point corresponding to the collapsed interval is an open singleton. Specifically, if $x \in (a,b)$ then $U = \{[x]\}$ is open since $[x] = (a,b)$ is open. Therefore $\mathbb{R}/\sim$ is not homeomorphic to $\mathbb{R}$.

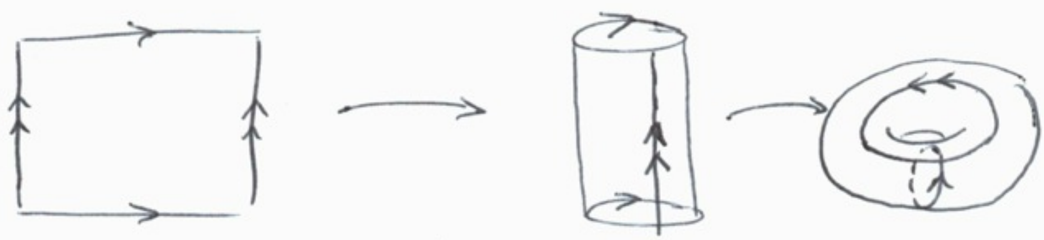
Example: Consider $[0,1]=X$ with equivalence relation $x \sim y$ if $x=y$ or $\{x,y\}=\{0,1\}$. Since $0 \sim 1$, the space $X/\sim$ is homeomorphic to $S^1 \subset \mathbb{R}^2$. An explicit homeomorphism is $h: X/\sim \longrightarrow S^1$, $h(x) = (\cos(2\pi x), \sin(2\pi x))$.

Example: Consider $[0,1]\times[0,1]$ with equivalence relation $(w,z)\sim(x,y)$ if $(w,z)=(x,y)$ or

$w=x$ and $\{z,y\}=\{0,1\}$ or

$z=y$ and $\{w,x\}=\{0,1\}$.

In pictures:



The resulting space is homeomorphic to $S^1 \times S^1$, with the product topology. It's a torus!

Proposition: Let X, Y be topological spaces and $f: X \to Y$ a map. Then define $\sim_f$ on X by $x_1 \sim x_2$ iff $f(x_1)=f(x_2)$.

(i) The mapping $g: X/\sim_f \longrightarrow Y$ defined by $g([x]) = f(x)$ is well-defined, one to one and continuous.

(ii) If f is open or closed, then g is a homeomorphism between $X/\sim_f$ and $f(X)$.

Proof: Clearly g is well defined, because if $x, y \in [z]$ then $f(x) = f(y) = f(z)$ by definition of $\sim_f$, also 1-1 is obvious. To check that g is continuous, first observe that

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[d, "q"'] \arrow[dr, "f"] & x \arrow[r, mapsto] & f(x) \\ {[x]} & X/\sim_f \arrow[r, "g"'] & Y & \\ & {[x]} \arrow[r, mapsto] & f(x). & \end{tikzcd}$$

commutes, so

$f = g \circ q$. Then let $U \subseteq Y$ be open. Since f is continuous, $f^{-1}(U)$ is open. But $f^{-1}(U) = q^{-1}(g^{-1}(U))$, so $q^{-1}(g^{-1}(U))$ is open. But then $q^{-1}(g^{-1}(U))$ is an open collection of equivalence classes, ie. $q^{-1}(g^{-1}(U)) = \bigcup_{[x] \in g^{-1}(U)} [x]$, thus $g^{-1}(U)$ is open by definition of the quotient topology. Therefore g is continuous, (i) follows.

(ii) Assume f is open, and $U \subset X/_{\sim_f}$ is open. Then $q^{-1}(U)$ is open and thefore $f(q^{-1}(U))$ is open. But $g(U) = f(q^{-1}(U))$, so $g(U)$ is open in Y. Therefore $g(U) \cap f(X)$ is open in $f(X)$, and $g: X/_{\sim_f} \to f(X)$ is a homeomorphism since it is a continuous open map.

Closed is similar, but take complements.

We can reword this proposition in terms of a <u>universal property</u>:

Given X with equivalence relation $\sim$ and quotient map $q: X \longrightarrow X/\sim$, the map $q: X \longrightarrow X/\sim$ is the unique map satisfying: If $f: X \longrightarrow Y$ is a continuous map such that $x \sim y$ implies $f(x) = f(y)$ $\forall\, x, y \in X$, then there exists a unique map $g: X/\sim \longrightarrow Y$ such that $f = g \circ q$.

I.e.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, "\exists ! g"'] & Y \end{tikzcd}$$

f agrees on equivalence classes.

$g([x]) = f(x)$.

The equivalence relation $\sim_f$ is "equivalence mod f".

<u>Example</u>: If $f: \mathbb{R} \longrightarrow \mathbb{R}$ is $f(x) = \lfloor x \rfloor$, then as a set, $\mathbb{R}/_{\sim_f}$ is $\mathbb{Z}$. What are its open sets?

Given $x \in \mathbb{R}$, its equivalence class is $[\lfloor x \rfloor, \lfloor x \rfloor + 1)$, which is not open. In fact, no set $\mathcal{U} \subset \mathbb{Z}$ can be open if it is bounded below. Otherwise, $\mathcal{U}$ contains a smallest integer n, and so

$$\bigcup_{[k] \in \mathcal{U}} [k] = \underbrace{[n, n+1)}_{\text{not open}} \cup \{\text{intervals contained in } [n+1, \infty)\}.$$

and thus $\mathcal{U}$ cannot be open.

The open sets are exactly
$$\{\mathcal{U} \subset \mathbb{Z} \mid n \in \mathcal{U} \text{ and } m < n \Rightarrow m \in \mathcal{U}\}.$$
E.g. $\mathcal{U} = \{\ldots -3, -2, -1, 0, 1, 2\}$
For then the set of equivalence classes in $\mathbb{R}$ is
$$\bigcup_{n \leq 2} [n, n+1) = (-\infty, 3), \text{ which is open.}$$
So $\mathbb{Z}$ has the nested topology, with open sets
$$\tau = \{(-\infty, n) \mid n \in \mathbb{Z}\}.$$

§4.3. The notion of quotient topologies allows us to start gluing spaces together.

Lemma (Gluing Lemma):

Let $A, B \subset X$ closed, with $X = A \cup B$ and $f : A \to Y$ and $g : B \to Y$ such that $g(x) = f(x)$ for all $x \in A \cap B$.
Then
$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}, \quad h : X \longrightarrow Y.$$
is continuous.

Proof: Check that $h : X \longrightarrow Y$ is continuous by showing $V \subset Y$ closed $\Rightarrow h^{-1}(V)$ closed.
First, $h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$. Now $f^{-1}(V)$ is closed (in A) and by definition of the subspace topology there exists F closed in $X = A \cup B$ such that $f^{-1}(V) = F \cap A$. But an ~~finite~~ intersection of closed sets is closed, so A closed $\Rightarrow f^{-1}(V) = F \cap A$ closed in X

Similarly we argue that $g^{-1}(V)$ is closed in B, hence closed in X.

Therefore $f^{-1}(V) \cup g^{-1}(V)$ is closed in X, too.