Topology 1 Jan 21.

Lecture 5

"Opposite" the the notion of density is

<u>Def</u>: A set $A \subset X$ is <u>nowhere dense</u> in X if $\mathrm{int}(\bar{A}) = \emptyset$.

<u>Example</u>: Any singleton $\{x\} \subseteq \mathbb{R}$ is nowhere dense. The integers $\mathbb{Z} \subseteq \mathbb{R}$ are also nowhere dense, since $\bar{\mathbb{Z}} = \mathbb{Z}$ and $\mathrm{int}(\bar{\mathbb{Z}}) = \emptyset$ (because $\mathbb{Z}$ contains no open sets).

<u>Example</u>: If $\mathbb{R}$ has the cofinite topology, then every finite set is nowhere dense. Why?

If $F \subseteq \mathbb{R}$ is finite, then it is closed, so $\bar{F} = F$. Then $\bar{F}$ contains no open sets since its complement is infinite, therefore $\mathrm{int}(\bar{F}) = \emptyset$.

<u>Def</u>: A subset $A \subset X$ is of the <u>first</u> <u>category</u> if it is a countable union of nowhere dense subsets. If A is not first category, then it is second category.

<u>Example</u>: $\mathbb{Q} \subseteq \mathbb{R}$ is first category, because

$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is a countable union. More bluntly, any countable subset of $\mathbb{R}$ is first category.

<u>Question</u>: Is $\mathbb{R}$ first or second category? (in itself)

Ans: It is second category.

<u>Rmk</u>: First category sets are sometimes called "meagre" and second category "abundant"

Theorem (Baire Category Theorem).

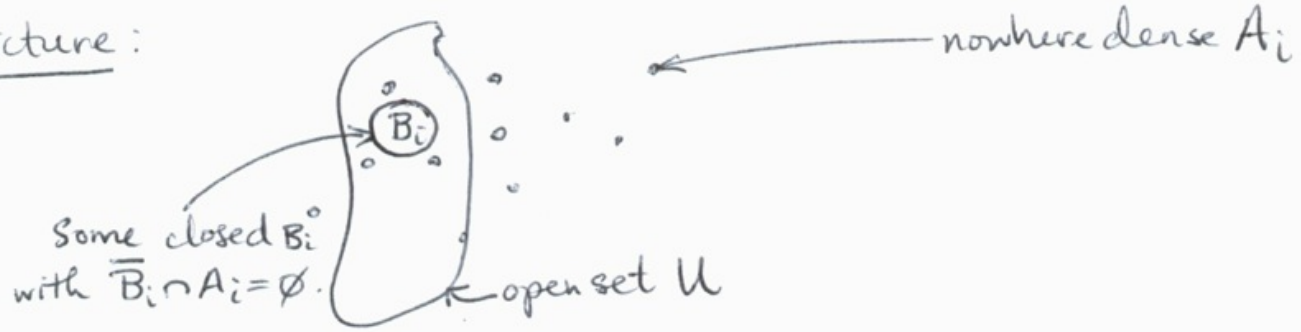
Every complete metric space is of the second category.

Proof: Let X be a complete metric space, for contradiction suppose X is first category - ie. suppose $X = \bigcup_{i=1}^{\infty} A_i$, where each A_i is nowhere dense.

To apply completeness we construct a Cauchy sequence, as follows.

Given $U \subset X$ open, for each A_i there is an open ball B_i with $\overline{B_i} \cap A_i = \emptyset$. (Exercise).

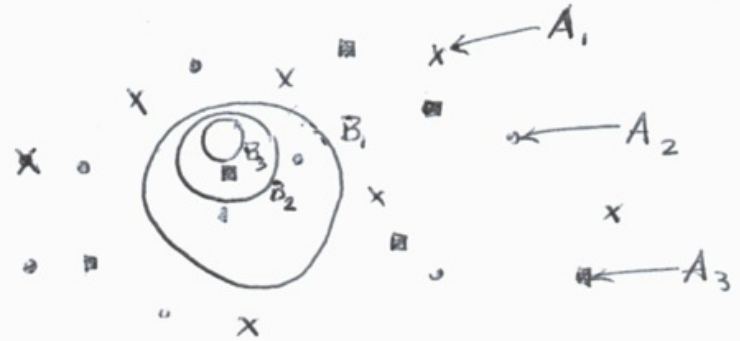
Picture:



However, we choose the B_i's a little more carefully than this.

- For B_1, choose any ball so that $\overline{B_1} \cap A_1 = \emptyset$.
- Choose $B_2 \subset B_1$ of radius $< \frac{1}{2}$, such that $\overline{B_2} \cap A_2 = \emptyset$.
- In general, choose $B_n \subset B_{n-1}$ of radius $< \frac{1}{n}$, such that $\overline{B_n} \cap A_n = \emptyset$.

Picture:

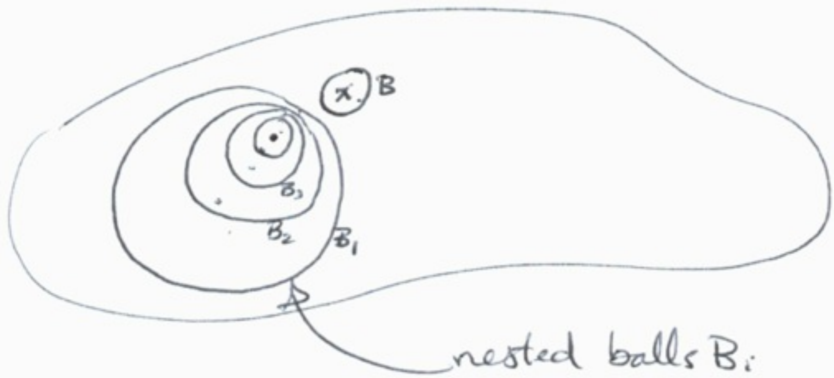


Let x_n denote the centre of B_n. Then $\{x_n\}$ is Cauchy, since $\{x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \subset B_n$, a ball of radius $\frac{1}{n}$. Thus x_n converges (X is complete), to $x_n \longrightarrow x$, say.

Observe that $x \in \overline{B}_n$ $\forall n$, because if not, say $x \notin \overline{B}_n$, then $\exists$ m such that $x \notin \overline{B}_m$ $\forall$ $m > n$.

But then $(\overline{B_n})^c$ is open, so there's an open ball B with $x \in B \in (\overline{B_n})^c$, contradicting the fact that $x_n \longrightarrow x$. (Because $x_m \notin B$ $\forall$ $m \geq n$).

Picture:



So $x \in \overline{B}_n$ $\forall n$. But this means $x \notin A_n$ for all n, since $\overline{B}_n \cap A_n = \emptyset$. A contradiction to the fact that $x \in X$, and $X = \bigcup_{i=1}^{\infty} A_i$.

Example: The set $\mathbb{R} \setminus \mathbb{Q}$ (irrationals) is second category. Why?

Since $\mathbb{Q}$ is first category (ie $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$), if $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{i=1}^{\infty} A_i$, A_i nowhere dense, then

$$\mathbb{R} = \bigcup_{i=1}^{\infty} A_i \cup \bigcup_{q \in \mathbb{Q}} \{q\}$$ would be first category, but it is not By Baire.

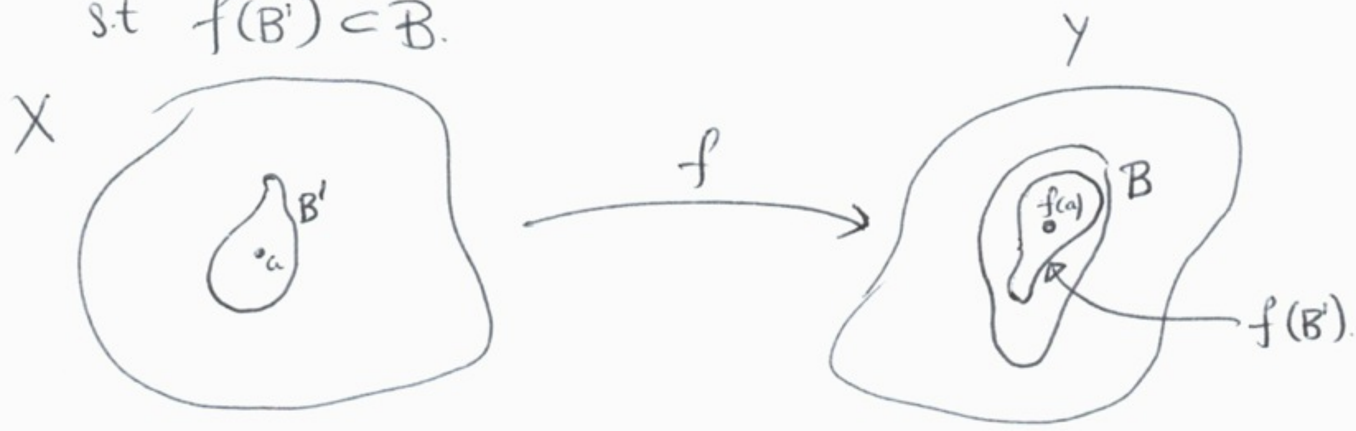
Example: Let (X,d) be a complete metric space without isolated points, i.e. no singleton $\{x\}$ is open. Then X is uncountable.

Proof: Since we are in a metric space, each $\{x_i\}$ is closed. Therefore $\overline{\{x_i\}} = \{x_i\}$. Since no singleton is open, $\operatorname{int}(\overline{\{x_i\}}) = \operatorname{int}(\{x_i\}) = \emptyset$ for all i. Thus $\{x_i\}$ is nowhere dense.
But $X = \bigcup_{i=1}^{\infty} \{x_i\}$, contradicting Baire's theorem.

§ 3.5 Continuous mappings

In analysis, you define continuous at a point first, then say something about global continuity. We'll do the same.

Def: Let X and Y be topological spaces.
A mapping $f: X \longrightarrow Y$ is continuous at $a \in X$ if $\exists$ local bases $\mathcal{B}_a$ and $\mathcal{B}_{f(a)}$ of a and $f(a)$ respectively, such that $\forall B \in \mathcal{B}_{f(a)}$ $\exists B' \in \mathcal{B}_a$ s.t $f(B') \subset B$.



Think in terms of balls in metric spaces:

$f: \mathbb{R} \to \mathbb{R}$ is cts at $x = a$ if $\forall\, B(f(a), \varepsilon)$ $\exists\, B(a, \delta)$ s.t. $f(B(a,\delta)) \subset B(f(a), \varepsilon)$.

Proposition 1: A mapping $f: X \to Y$ is continuous at $a \in X$ iff for every open neighbourhood V of $f(a)$ $\exists$ a nbhd U of a such that $f(U) \subseteq V$.

Proof: $(\Rightarrow)$ Suppose f cts at a, and let $\mathcal{B}_a$, $\mathcal{B}_{f(a)}$ be local bases as in the definition. Let V be an open nbhd of $f(a)$, choose $B_{f(a)} \subset V$ by def of a local basis. Since f cts, $\exists$ $B_a \in \mathcal{B}_a$ st. $f(B_a) \subset B_{f(a)} \subset V$. Take $U = B_a$.

$(\Leftarrow)$. Take $\mathcal{B}_a = \{\text{all open nbhds of } a\}$

$\mathcal{B}_{f(a)} = \{\text{all open nbhds of } f(a)\}$,

and it follows that f is cts.

$\boxed{\text{We say } f \text{ is continuous if it is cts at every } a \in X}$

Proposition: Let X and Y be two spaces and $f: X \to Y$ a mapping. The following are equivalent:

(a) $f: X \to Y$ is continuous.

(b) If $U \subset Y$ is open, then $f^{-1}(U)$ is open in X.
(in Y)

(c) For every basis $\mathcal{B}_Y$ of Y, and every $B \in \mathcal{B}_Y$, the set $f^{-1}(B_Y)$ is open.

Remark: We mostly use (b) as the definition of continuity.

Proof: We do (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b)

If f is continuous and $U \subset Y$ is open, then every $a \in f^{-1}(U)$ has a nbhd V_a s.t. $f(V_a) \subset U$. I.e. $V_a \subset f^{-1}(U)$. But then $f^{-1}(U) = \bigcup_{a \in f^{-1}(U)} V_a$, so it's open.

(b) $\Rightarrow$ (c) Is true since basis sets are open sets.

(c) $\Rightarrow$ (a) Assume $\mathcal{B}_Y$ is a basis so that every $B \in \mathcal{B}_Y$ satisfies $f^{-1}(B)$ is open.

Set $\mathcal{B}_{f(a)} = \{ B \in \mathcal{B}_Y \mid f(a) \in B \}$, a local basis at $f(a)$

Let V be an open nbhd of $f(a)$, then there exists B s.t. $f(a) \in B \subset V$. Then $f^{-1}(B)$ is an open nbhd of a s.t. $f(f^{-1}(B)) \subset V$, so by previous (proposition 1) f is cts.

Proposition: A map $f: X \to Y$ is continuous if and only if inverse images of closed sets are closed.

Topology 1 January 23
Lecture 6.

<u>Proposition</u>: Suppose that X is first countable and Y is any space. A map $f: X \longrightarrow Y$ is continuous at $a \in X$ iff for every sequence $\{x_n\}$ converging to a the sequence $\{f(x_n)\}$ converges to $f(a)$.

<u>Proof</u>: ($\Rightarrow$) Suppose that $x_n \longrightarrow a$ and f is cts. Let $V \subset Y$ be any nbhd of $f(a)$. Since f is cts at a $\exists\, \mathcal{U}$ with $a \in \mathcal{U}$ s.t. $f(\mathcal{U}) \subset V$. Since $x_n \longrightarrow a$, $\mathcal{U}$ contains x_m $\forall m \geq N$ (some N). But then $f(x_m) \in f(\mathcal{U}) \subset V$ $\forall m \geq N$, so $f(x_n) \longrightarrow f(a)$. (Note didn't use X first countable).

($\Leftarrow$).

Suppose that for every $\{x_n\}$ converging to a in X, $\{f(x_n)\}$ converges to $f(a)$ in Y.

Let V be a nbhd of $f(a)$, we find $\mathcal{U}$ s.t. $f(\mathcal{U}) \subset V$. We will do this by showing that $a \in \operatorname{int}(f^{-1}(V))$.

Suppose a is <u>not</u> an interior point, and let $\mathcal{B}_a = \{B_i \mid i \in \mathbb{Z}^+\}$ be a countable local basis at a.

The set $\bigcap_{i=1}^{n} B_i$ is open and contains a $\forall n \in \mathbb{Z}^+$.

Since a is <u>not</u> an interior point of $f^{-1}(V)$,

typo? $\left(\bigcap_{i=1}^{n} B_i\right) \cap \left(f^{-1}(V)\right)^{c} \neq \phi$, if it were empty then $a \in \bigcap_{i=1}^{n} B_i \subset f^{-1}(V)$ and a is interior. So, choose $x_n \in \left(\bigcap_{i=1}^{n} B_i\right) \cap f^{-1}(V)$ for each i, which defines a sequence $\{x_n\}$ with $x_n \longrightarrow x$ by construction.

By assumption, then $f(x_n) \longrightarrow f(a)$, by construction all $f(x_n)$'s are outside of V. Contradiction.

<u>Example</u>: (First countability is necessary).

Give $\mathbb{R}$ the co-countable topology, ie

$\tau = \{U \subseteq \mathbb{R} \mid U^c \text{ is countable}\}$.

<u>Exercise</u>: With this topology, $\mathbb{R}$ is not first countable.

A sequence $\{x_n\}$ in $\mathbb{R}$ converges in this topology if and only if it is eventually constant; ie $\exists N$ s.t. $x_N = x_{N+1} = x_{N+2} \cdots = x_{N+k} = \cdots \quad \forall k > 1$.

In this case x_n converges to x_N.

Give the integers the cofinite topology and define $f: \mathbb{R} \longrightarrow \mathbb{Z}$ to be the floor function,

$x \longmapsto \lfloor x \rfloor$ = the greatest integer less than x.

Now observe:

If $x_n \longrightarrow x$ in $\mathbb{R}$, then it is eventually constant, ie. $x_m = x_N = x \ \forall\ m \geq N$.

Then $\{f(x_n)\}$ is also eventually constant, since $f(x_m) = f(x_N)\ \forall m \geq N$. Therefore $f(x_n) \longrightarrow f(x_N) = f(x)$.

On the other hand, f is not continuous:

The open set $V = \mathbb{Z} \setminus \{0\}$ has preimage $f^{-1}(V) = \mathbb{R} \setminus [0,1)$, a set with uncountable complement. Therefore $f^{-1}(V)$ is not open.

<u>Def</u>: A map $f: X \longrightarrow Y$ is open if for every open $U \subseteq X$, the set $f(U)$ is open. A map $f: X \to Y$ is closed if for every closed subset $F \subseteq X$, $f(F)$ is closed.

<u>Rmk</u>: If $f: X \to Y$ is bijective, then

open $\Rightarrow f^{-1}$ continuous

closed $\Rightarrow f^{-1}$ continuous.

ie. For bijections, f is open $\Longleftrightarrow f$ is closed.

<u>Definition</u>: A map $f: X \longrightarrow Y$ is a homeomorphism if it is an open, continuous bijection (ie. f and f^{-1} are continuous).

If $\exists$ a homeomorphism $f: X \to Y$, we say X and Y are homeomorphic and write $X \simeq Y$.
(Homeomorphisms are the isomorphisms of topology).

Properties preserved by homeomorphisms are called <u>topological properties</u>.

<u>Example</u>: $f:(0,\infty) \to \mathbb{R}$ given by $f(x) = \ln(x)$ is a homeomorphism,

<u>Example</u>: Let $\varphi: \mathbb{Z}\times\mathbb{Z} \xrightarrow{\text{bijection}} \mathbb{Z}^+$ be any enumeration of $\mathbb{Z}\times\mathbb{Z}$, such a map φ exists since $\mathbb{Z}\times\mathbb{Z}$ is countable. Give both $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}^+$ the cofinite topology. Then because φ is a bijection,

$U \subset \mathbb{Z}\times\mathbb{Z}$ has finite complement $\Leftrightarrow$ $\varphi(U)$ has finite complement.

ie $\quad U$ open $\Leftrightarrow f(U)$ is open, thus $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}^+$ are homeomorphic.

More generally, if X and Y have the cofinite topology, then $X \simeq Y$ iff there exists a bijection $\varphi: X \to Y$.

<u>Example</u>: If two spaces have different cardinality, they cannot be homeomorphic.

Some examples of topological properties:

<u>Proposition</u>: Let $f: X \longrightarrow Y$ be a homeomorphism. Then

(i) If X is first countable, so is Y.

(ii) If X is second countable, so is Y.

<u>Proof</u>: (i) Will be left as an exercise (maybe next asst).

(ii) If X is second countable, then there's a countable basis $\mathcal{B}_X$. Set $\mathcal{B}_Y = \{f(B) \mid B \in \mathcal{B}_X\}$, we'll show $\mathcal{B}_Y$ is a basis for Y (it is certainly countable), since f is a homeomorphism $\mathcal{B}_Y$ consists of open sets. Let $V \subset Y$ be open, ~~Then $f^{-1}(V)$ is open, so $\exists B \in \mathcal{B}_X$ st.~~ and let $y \in Y$ be given. Then $f^{-1}(V)$ is open and contains the point $f^{-1}(y)$. Since $\mathcal{B}_X$ is a basis $\exists B \in \mathcal{B}_X$ st. $f^{-1}(y) \in B \subset f^{-1}(V)$. But then $f(B) \in \mathcal{B}_Y$ satisfies $y \in f(B) \subset V$, so $\mathcal{B}_Y$ is a basis.

Some properties of subsets can also be preserved:

<u>Proposition</u>: If $f: X \longrightarrow Y$ is a homeomorphism and $A \subset X$, then:

(i) $f(\mathrm{int}(A)) = \mathrm{int}(f(A))$

(ii) $f(\bar{A}) = \overline{f(A)}$

(iii) $f(\partial A) = \partial(f(A))$

(iv) $f(A') = (f(A))'$

Proof:

We'll prove only one of them (proofs like this can be long).

Proof of (i)

Let $x \in f(\mathrm{int}(A))$. Then $f^{-1}(x) \in \mathrm{int}(A)$, so $\exists\, U$ open in X s.t. $f^{-1}(x) \in U \subset A$. But then applying f gives $x \in f(U) \subset f(A)$, where $f(U)$ is open since f is a homeo. Thus $x \in \mathrm{int}(f(A))$. So $f(\mathrm{int}(A)) \subset \mathrm{int}(f(A))$.

On the other hand suppose $x \in \mathrm{int}(f(A))$. Then $\exists\, V$ open s.t. $x \in V \subset f(A)$. So $f^{-1}(x) \in f^{-1}(V) \subset A$, where $f^{-1}(V)$ is open since f is a homeomorphism. Thus $f^{-1}(x) \in \mathrm{int}(A)$, so $x \in f(\mathrm{int}(A))$. Thus $\mathrm{int}(f(A)) \subset f(\mathrm{int}(A))$.

Therefore $\mathrm{int}(f(A)) = f(\mathrm{int}(A))$.