Topology 1 January 14

Lecture 3.

Recall that we just defined the set of accumulation points A' of a set A, and stated:

Theorem: (a) If $A \subset B$, then $A' \subset B'$ and
(b) For every subset A of a space X we have $\bar{A} = A \cup A'$

(Here $\bar{A}$ is closure). Def: If $\{x_n\} \subset X$, $\lim_{n\to\infty} x_n = x$ if $\forall$ open U with $x \in U$, $\exists N$ s.t. $n > N \Rightarrow x_n \in U$.

Definition: Let $A \subset X$. Then $x \in X$ is a boundary point of A if every open neighbourhood of x intersects both A and A^c nontrivially.

Example: In $\mathbb{R}^2$, the boundary of the subset $(0,1] \times (0,1]$ is exactly what you expect:

Example: Give $\mathbb{R}$ the cofinite topology, and let $A \subseteq \mathbb{R}$ be an infinite set with infinite complement. (E.g. $(a,b) \subseteq \mathbb{R}$ with a, b finite).
Given $x \in \mathbb{R}$, every open set containing x intersects both A and A^c nontrivially, so x is a boundary point of A.

Notation: Boundary of A is written ∂A, here $\partial A = \mathbb{R}$.

Theorem: For every $A \subset X$, the following are true:

(a) ∂A is closed.

(b) $\bar{A} = \operatorname{int} A \cup \partial A$ and $\operatorname{int} A \cap \partial A = \emptyset$.

(c) A is clopen iff $\partial A = \emptyset$.

Proofs (partial).

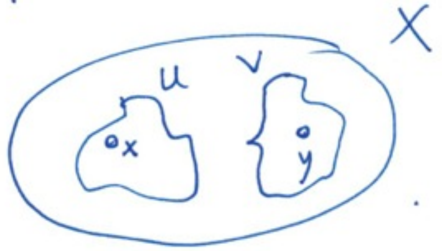
a) Observe $X \setminus \partial A = (A \setminus \partial A) \cup (A^c \setminus \partial A)$, we show both of these sets are open. Since every point $x \in A \setminus \partial A$ is not in ∂A, $\exists$ U open s.t. $x \in U$ and $U \cap A^c = \emptyset$; it follows that $U \cap \partial A = \emptyset$ also (by def of ∂A).
Thus $U \subset A \setminus \partial A$, so x is an interior point. Therefore $\operatorname{int}(A \setminus \partial A) = A \setminus \partial A$ is open, by symmetry so is $A^c \setminus \partial A$.

c), b) Technical definition-checking.

Last, we introduce a definition:

Def: A space X is Hausdorff if for every distinct pair of points $x, y \in X$, there are disjoint open neighbourhoods $x \in U$ and $y \in V$.

Picture:

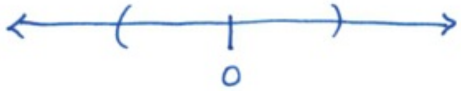


Example: Let $X = \mathbb{R} \cup \{z\}$. Define a topology on X as follows. Let $\mathcal{T}$ denote the "typical" ~~regular~~ topology on $\mathbb{R}$.
Set $\mathcal{T}' = \mathcal{T} \cup \{(U \setminus \{0\}) \cup \{z\} \mid U \in \mathcal{T} \text{ and } 0 \in U\}$.

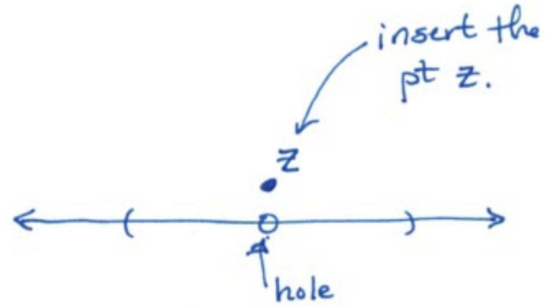
Here, the point "z" is a second zero.

~~Sk~~ Schematically:

A typical interval about 0:



becomes



Then z, 0 do not have disjoint neighbourhoods, so X is not Hausdorff. (Why? Because every open nbhd of 0 <u>and</u> every open nbhd of z must contain a set of the form $(-\varepsilon, \varepsilon) \setminus \{0\}$).

<u>§ 3.3</u> Bases.

<u>Definition</u>.

Given a topological space $(X, \mathcal{T})$, a <u>basis</u> for the topology $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X such that:

(i) Every $B \in \mathcal{B}$ is open.

(ii) Every $U \in \mathcal{T}$ is a union of sets in $\mathcal{B}$.

<u>Proposition 1</u> Let $\mathcal{B}$ be a collection of open sets. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if $\forall U \in \mathcal{T}$ and $x \in U$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$. (Every $x \in U$ has a $\mathcal{B}$-nbhd in U)

<u>Proof</u>:

($\Rightarrow$) If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\forall x \in U$ open $\exists B$ with $x \in B \subset U$, since U is a union of elements of $\mathcal{B}$.

($\Leftarrow$) Given $U \in \mathcal{T}$, $\forall x \in U$ $\exists B_x \subset U$ with $x \in B_x \subset U$.

Therefore $U = \bigcup_{x \in U} B_x$, so $\mathcal{B}$ is a basis.

Example: The usual topology on $\mathbb{R}$ has many bases, here are some.

(i) $\{(a,b) \mid a,b \in \mathbb{R}\}$

(ii) $\{(a,b) \mid a,b \in \mathbb{Q}\}$

(iii) $\left\{ \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \middle| \; q_1, q_2 \text{ are multiples of } 2 \right\}$.

i.e. Bases are not unique, there is no preferred basis

Example: The open balls in a metric space form a basis for the topology.

Example: The divisor topology on $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Define $\mathcal{T}$ to be the set of $U \subset \mathbb{Z}^+$ satisfying:

If $n \in U$ and x divides n, then $x \in U$.

Then a basis for $\mathcal{T}$ is given by the sets:

$$S_n = \{x \in \mathbb{Z}^+ \mid x \text{ divides } n\}.$$

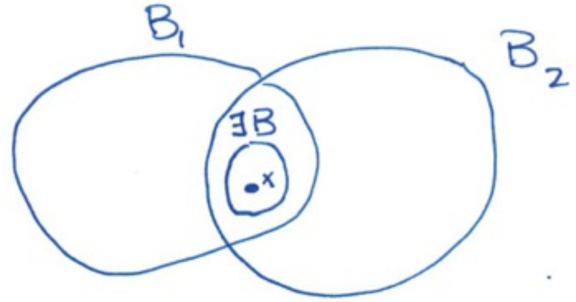
Why? Because given $U = \{n_1, n_2, n_3 \ldots\} \in \mathcal{T}$, we can write $U = \bigcup_{i=1}^{\infty} S_{n_i}$, in particular $\forall n \in U$ $n \in S_n \subset U$.

Theorem: A collection $\mathcal{B}$ is a basis for some topology on X iff

(i) $\bigcup_{B \in \mathcal{B}} B = X$

(ii) For every pair $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

E.g.



Proof:

($\Longrightarrow$) Suppose $\mathcal{B}$ is a basis for some $\mathcal{T}$.
Then $X \in \mathcal{T}$, so $\exists \{B_i\}_{i \in I} \subset \mathcal{B}$ s.t. $\bigcup_{i \in I} B_i = X$, so (i) is true.
Given $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2 \in \mathcal{T}$. Thus $\forall x \in B_1 \cap B_2$ $\exists B$ s.t. $x \in B \subset B_1 \cap B_2$, by previous (proposition 1).

($\Longleftarrow$) Suppose $\mathcal{B}$ satisfies (i) and (ii), and let $\mathcal{T}$ denote all unions of elements of $\mathcal{B}$. If $\mathcal{T}$ is a topology, then $\mathcal{B}$ will be a basis for it and we're done.
$\mathcal{T}$ obviously satisfies (i) and (ii) in the definition of a topology (we take $\emptyset$ = the empty union). So we need to show finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.

We start with $U_1, U_2 \in \mathcal{T}$. Write
$U_1 = \bigcup_{i \in I} B_i$ and $U_2 = \bigcup_{j \in J} B_j$. Then

$$U_1 \cap U_2 = \left(\bigcup_{i \in I} B_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{\substack{i \in I \\ j \in J}} (B_i \cap B_j).$$

So if each $B_i \cap B_j$ is a union of elements of $\mathcal{B}$, then $U_1 \cap U_2 \in \mathcal{T}$.
However, this follows from assumption (ii) in the hypotheses of the theorem. Thus $U_1 \cap U_2 \in \mathcal{T}$.
For arbitrary finite intersections, the claim follows by induction.

$\blacksquare$

~~Exa~~ Definition: For a set $\mathcal{B}$ of subsets satisfying (i) and (ii) above, the set of all unions of elements of $\mathcal{B}$ is called the topology generated by $\mathcal{B}$.

END

Example: The evenly-spaced topology on $\mathbb{Z}$ is the topology generated by the basis $\mathcal{B}$ consisting of sets of the form

$$S(a,b) = \{ na+b \mid n \in \mathbb{Z} \} \quad a \neq 0.$$

The set $\mathcal{B} = \{ S(a,b) \mid a,b \in \mathbb{Z} \}$ obviously satisfies (i) $\bigcup_{B \in \mathcal{B}} B = \mathbb{Z}$ and (ii)

(ii) Consider $S(a_1,b_1) \cap S(a_2,b_2)$. The intersection consists of all integers x satisfying

$$x \equiv b_1 \bmod a_1 \qquad x \equiv b_2 \bmod a_2$$

By the Chinese remainder theorem, there is a solution for x only when $b_1 = b_2 \ (\mathrm{mod}\ \gcd(a_1,a_2))$ and in this case the solution is unique mod $\mathrm{lcm}(a_1,a_2)$.

I.e, either $S(a_1,b_1) \cap S(a_2,b_2) = \emptyset \in \mathcal{T}$ or $\exists\, x$ s.t.

$$S(a,x) \subset S(a_1,b_1) \cap S(a_2,b_2) \text{ where } a = \mathrm{lcm}(a_1,a_2).$$

Definition: If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X and $\mathcal{T}_1 \subset \mathcal{T}_2$, we say $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

Proposition: If $\mathcal{B}$ generates the topology τ on X and $\mathcal{B} \subset \tau'$, then $\tau \subset \tau'$.
In other words, ~~$\mathcal{B}$~~ τ is the coarsest topology containing $\mathcal{B}$.

Example (Sorgenfrey line).
The real line $\mathbb{R}$ with τ' generated by

$$\{[a,b) \mid a, b \in \mathbb{R}\}$$

will be called the Sorgenfrey line.
Since $(c,b) = \bigcup_{c<a} [a,b)$, the 'normal' open intervals (c,b) are open in the Sorgenfrey topology as well.
In other words, if τ is the usual topology on $\mathbb{R}$ we have $\tau \subset \tau'$, so the Sorgenfrey line has a finer topology than $\mathbb{R}$.

Topology 1 January 16
Lecture 4.

<u>Definition</u>: A space (X, τ) is <u>second countable</u> if it has a countable basis.

<u>Example</u>: We already saw that $\mathbb{R}$ has basis $\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{Q}\}$, which is countable since $\mathbb{Q}$ is countable.

<u>Example</u>: The Sorgenfrey line is not second countable.

Let $I = \{[x, x+1) \mid x \in \mathbb{R}\}$. The set I is not countable, and every set in I is open.
Suppose $\mathcal{B}$ is a basis for the Sorgenfrey line. Then $\forall [x, x+1) \in I$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset [x, x+1)$. To show $\mathcal{B}$ is uncountable, we need only show that $B_x \neq B_y$ whenever $x \neq y$.

So suppose WLOG $x < y$. Then $x \notin [y, y+1)$ ~~yet~~ and so $x \notin B_y$, yet $x \in B_x$. Thus B_x and B_y are distinct elements of $\mathcal{B}$ $\forall x, y \in \mathbb{R}$, so $\mathcal{B}$ is uncountable.

<u>Definition</u>: A <u>local basis</u> of a point $x \in X$ is a set $\mathcal{B}_x$ of nbhds of x s.t. for all open sets U with $x \in U$, $\exists B \in \mathcal{B}_x$ s.t. $x \in B \subset U$.

<u>Picture</u>:

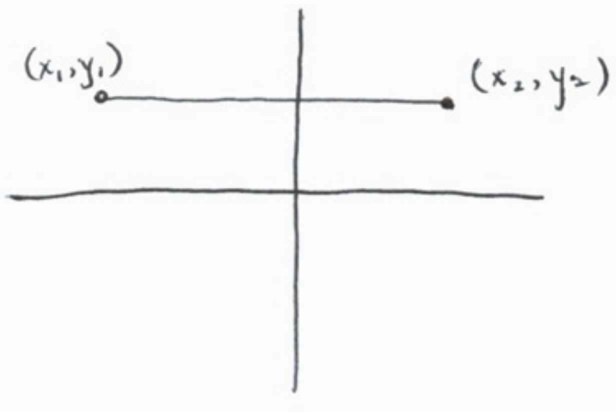
Proposition: If $\mathcal{B}$ is a basis of (X, τ), then $\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$ is a local basis at x.

Proposition: If $\{\mathcal{B}_x\}_{x \in X}$ is a collection of local bases, one for each point of X, then $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a basis for X.

Definition: A space X is first countable if every $x \in X$ has a countable local basis.

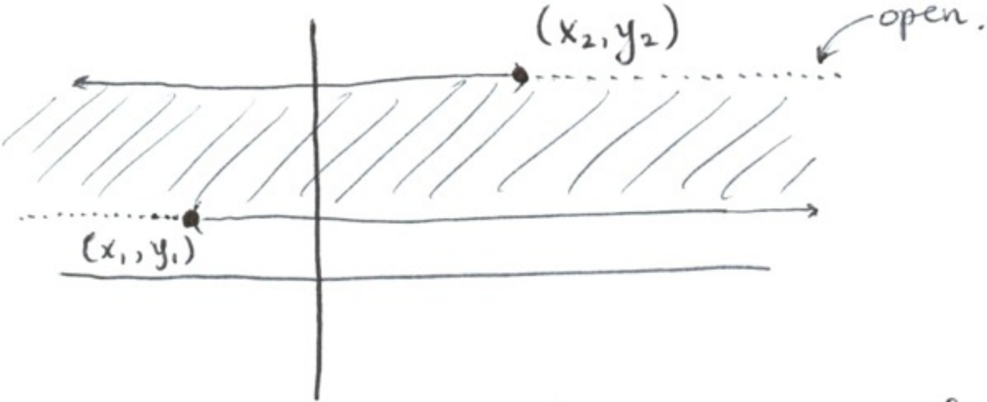
Example: Metric spaces are first countable.
For each $x \in X$, with metric $d(x,y)$, set $\mathcal{B}_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{Z}^+$, open balls$\}$. Then $\mathcal{B}_x$ is a countable local basis at x.

Example: Consider $\mathbb{R}^2$, and order it as follows: declare $(x_1, y_1) < (x_2, y_2)$ if $y_1 < y_2$ or $y_1 = y_2$ and $x_1 < x_2$.
Give $\mathbb{R}^2$ the order topology. So open sets look like:



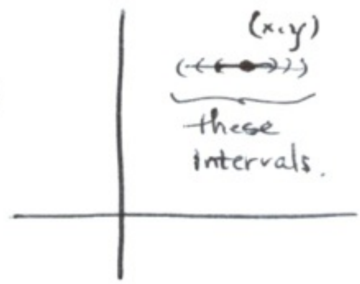
(if $y_1 = y_2$)

if $y_1 < y_2$



This is the lexicographic order topology.

Then $\mathbb{R}^2$ with this topology is first countable, because given $(x,y) \in \mathbb{R}^2$ the intervals $\left(\left(x-\frac{1}{n}, y\right), \left(x+\frac{1}{n}, y\right)\right)$ form a local basis:



<u>Definition</u>: A collection S of open subsets of (X, τ) is a subbasis for τ if the set of finite intersections of elements of S form a basis. I.e

$\mathcal{B}_S = \{U_1 \cap \dots \cap U_k \mid U_i \in S,\ k \in \mathbb{Z}\}$ is a basis.

<u>Remark</u>: A collection S is a subbasis for some topology iff $\mathcal{B}_S$ satisfies (i) and (ii) of the basis theorem, ie.

(i) $\bigcup_{B \in \mathcal{B}} B = X$

(ii) $\forall B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ $\exists B \in \mathcal{B}$ s.t $x \in B \subset B_1 \cap B_2$.

However, since $\mathcal{B}_S$ contains all finite intersections (ii) is automatic. So we only need (i) in order to get a topology from S.

<u>Prop</u>: S is a subbasis for some topology τ iff $\bigcup_{U \in S} U = X$. In this case, τ is said to be generated by S.

<u>§3.4</u> Density.

<u>Definition</u>: A subset $D \subset X$ is dense in X if $\forall$ nonempty open $U \subset X$, $U \cap D \neq \emptyset$.

<u>Example</u>: A set is always dense in itself.

<u>Example</u>: $\mathbb{Q}^n$ is always dense in $\mathbb{R}^n$ for every n.

<u>Example</u>: Give $\mathbb{R}$ the cofinite topology. Then every infinite subset of $\mathbb{R}$ is dense in $\mathbb{R}$.

<u>Proposition</u>: D is dense in X iff $\overline{D} = X$.

<u>Proof</u>. ($\Rightarrow$) Suppose D is dense. Recall that $\overline{D} = D \cup D'$. By definition of density, every $x \in X \setminus D$ is an accumulation point of D, so done.

($\Leftarrow$) Suppose $\overline{D} = X$, and let $U \subseteq X$ be open. Given $x \in U$ either $x \in D$ and $U \cap D \neq \emptyset$, or $x \in D'$ and by definition of accumulation points $D \cap U \neq \emptyset$.

<u>Def</u>: A space X is separable if X has a countable dense subset.

<u>Proposition</u>: Every second countable space is separable.

<u>Proof</u>: Let $\mathcal{B}$ be a countable basis for X.
For each $B \in \mathcal{B}$, choose $x_B \in B$. Then $\{x_B\}_{B \in \mathcal{B}}$ is countable, and it's dense because every open U is a union of elements of $\mathcal{B}$, so some x_B is in U.

<u>Proposition</u>: Every separable metric space is second countable.

<u>Proof</u>: Let (X, d) be a metric space with $D \subset X$ countable and dense.

Set $\mathcal{B} = \{B(x, \frac{1}{n}) \mid x \in D \text{ and } n \in \mathbb{Z}^+\}$, we show $\mathcal{B}$ is a basis. ($\mathcal{B}$ is countable because there one ball for each $(x, n) \in D \times \mathbb{Z}^+$, a product of countable sets thus countable).

Let $U \subset X$ open, and $y \in U$ any point. Then $\exists\, \varepsilon > 0$ such that $B(y, \varepsilon) \subset U$. Choose $n > \frac{2}{\varepsilon}$, so $\frac{2}{n} < \varepsilon$ and $B(y, \frac{2}{n}) \subset B(y, \varepsilon)$.

But now D is dense, so $\exists\, x \in D \cap B(y, \frac{1}{n})$, hence $y \in B(x, \frac{1}{n})$.

Now we check that $B(x, \frac{1}{n}) \subset B(y, \frac{2}{n})$. This follows from $\forall z \in B(x, \frac{1}{n})$:

$$
\begin{aligned}
d(z, y) &\leq d(z, x) + d(x, y) \\
&\leqslant \frac{1}{n} + \frac{1}{n} \\
&= \frac{2}{n}
\end{aligned}
$$

So $\underset{y\in}{}\overset{\mathcal{B}}{\underset{\cup}{}}B(x,\tfrac{1}{n}) \subset B(y,\tfrac{2}{n}) \subset B(y,\varepsilon) \subset \mathcal{U}$.

Thus $\mathcal{B}$ is a basis.

Picture: