MATH 3240 April 8.

Today we finish our investigation of separation axioms, by investigating regularity and normality.

Previously stated in haste, now stated correctly:

<u>Theorem</u>: If X is a T_i-space, then every subset of X is a T_i-space for $i=1,2,3$ (not 4!)

<u>Goal</u>: Provide a counterexample when $i=4$.

<u>Theorem</u>: There exists a normal space X with a subspace A that is not normal.

<u>Theorem</u>: Every compact Hausdorff space is normal.

<u>Proof</u>: First we show that if X is Hausdorff and compact, then it is regular.

Let $x \in X$ and $A \subset X$ closed be given. Then $\forall y \in A$, choose disjoint U_y and V_y with $x \in U_y$ and $y \in V_y$. Then A is compact and $A \subset \bigcup_{y \in A} V_y$, so there's a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$ of A.

Set $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$. Both are open, disjoint by construction, and $A \subset V$, $x \in U$. So we have regularity.

For normality, let $A, B \subset X$ closed. For every $x \in B$, there are disjoint open U_x, V_x with $x \in U_x$ and $A \subset V_x$, by regularity.

Then B is compact and $B \subset \bigcup_{x \in B} U_x$, so let $\{U_{x_1}, \dots, U_{x_m}\}$ be a finite subcover. Then set $U = \bigcup_{i=1}^{m} U_{x_i}$, $V = \bigcap_{i=1}^{m} V_{x_i}$. Both are open, disjoint by construction, and $B \subset U$, $A \subset V$. Thus X is normal.

A strange non-normal space.

Let $\mathbb{Z}^+$ denote the positive integers with the discrete topology, and take one copy $\mathbb{Z}^+_r$ for each $r \in \mathbb{R}$. Consider the product $X_{\mathbb{R}} = \prod_{r \in \mathbb{R}} \mathbb{Z}^+_r$.

<u>Claim</u>: $X_{\mathbb{R}}$ is not normal. (This proof is hard).

For $i = 0, 1$ define a set $P_i \subset X_{\mathbb{R}}$ as follows: P_i is the collection of all points with the property that each integer except i appears <u>at most once</u> as a coordinate. So for example, elements of P_0 can have 0 in infinitely many coordinates but 2 can only appear as a coordinate at most once.

Then $P_0 \cap P_1 = \emptyset$ since $\mathbb{Z}^+$ is countable yet elements of $X_{\mathbb{R}}$ have uncountably many coordinates. Moreover, each P_i is a closed set because we can write

$$X_{\mathbb{R}} \setminus P_i = \bigcup_{\substack{r \neq s \\ n \neq i \\ (r, s \in \mathbb{R})}} \left(p_r^{-1}(n) \cap p_s^{-1}(n) \right)$$

which is a union of sets which are open in the product topology.

Note: Each set in the union consists of all sequences with some integer 'n' appearing at least twice

We will show that even though the P_i are closed, they cannot be separated by open sets. Thus $X_{\mathbb{R}}$ is not normal. Suppose $P_0 \subset U$, and $P_1 \subset V$.

If $F \subseteq \mathbb{R}$ is a finite subset, for each $x \in X_{\mathbb{R}}$ there is an associated basic open set $F(x) = \bigcap_{r \in F} p_r^{-1}(x_r)$, $p_r^{-1}(x_r)$ is all sequences with x_r in the r^{th} coordinate.

Define a nested increasing sequence $F_n = \{r_j\}_{j=1}^{n}$ of finite subsets of $\mathbb{R}$, together with an associated sequence of points $x^n \in P_0$ as follows.

Let $x_r^0 = 0$ for all $r \in \mathbb{R}$, so $x^0 \in X_{\mathbb{R}}$ is just a sequence of zeroes. Now suppose x^n and F_{n-1} are given, choose $F_n \supset F_{n-1}$ so that $F_n(x^n) \subset U$ (this is possible but may require that we add more than one point to F_{n-1}) and choose $x^{n+1} \in P_0$ so that $x_{r_j}^{n+1} = j$ whenever $r_j \in F_n$, and $x_r^{n+1} = 0$ otherwise.

Now define $y \in P_1$ by $y_{r_j} = j$ whenever $r_j \in \bigcup F_n$, and $y_r = 1$ otherwise.

Then there is a finite set $G \subseteq R$ such that $G(y) \subset V$. For some integer m,

$$G \cap \left(\bigcup_{n=0}^{\infty} F_n \right) = G \cap F_m,$$

so we can define $z \in X_{\mathbb{R}}$ by $z_{r_k} = k$ whenever $r_k \in F_m$, $z_{r_k} = 0$ whenever $\alpha_k \in F_{m+1} - F_m$, and $z_r = 1$ otherwise.

Then by construction, $z \in U \cap V$, which we check: By our choices, $z_r = y_r$ if $r \in G \cap F_m$; otherwise if $r \in G$ then $z_r = y_r = 1$; therefore $z \in G(y) \subset V$. Further $z_{r_k} = k = x_{r_k}^{m+1}$ if $r_k \in F_m$ and $z_{r_k} = 0 = x_{r_k}^{m+1}$ if $r_k \in F_{m+1} - F_m$, so

$$z \in \bigcap_{r \in F_{m+1}} \left(p_r^{-1}(x_r^{m+1}) \right) = F_{m+1}(x^{m+1}) \subset U. \text{ So } z \in U \cap V.$$

Now we can construct a non-normal subspace of <u>a</u> normal space:

Observe that there is a continuous embedding $\mathbb{Z}^+ \hookrightarrow \mathbb{R}$, and so we can construct an embedding

$$X_{\mathbb{R}} \hookrightarrow \prod_{r \in \mathbb{R}} \mathbb{R}$$

which is the map defined by $i \longmapsto i$ in each coordinate. Since $\mathbb{R} \simeq (0,1)$, there is then a homeomorphism $\prod_{r\in\mathbb{R}} \mathbb{R} \simeq \prod_{r\in\mathbb{R}} (0,1)$, and evidently another embedding $\prod_{r\in\mathbb{R}}(0,1) \hookrightarrow \prod_{r\in\mathbb{R}}[0,1]$. Thus

$$f: X_{\mathbb{R}} \hookrightarrow \prod_{r\in\mathbb{R}} \mathbb{R} \simeq \prod_{r\in\mathbb{R}}(0,1) \hookrightarrow \prod_{r\in\mathbb{R}}[0,1].$$

The space $\prod_{r\in\mathbb{R}}[0,1]$ is Hausdorff, and by Tychonoff's theorem it is compact. Thus it is normal, but the subspace $f(X_{\mathbb{R}})$ is not.

<u>END COURSE</u>