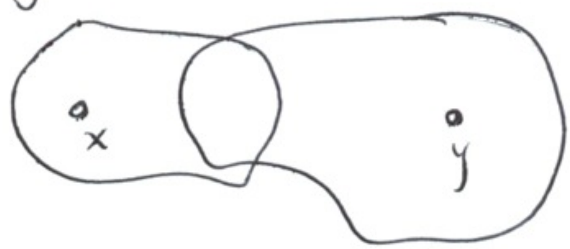
MATH 3240 Topology

<u>Def</u>: A space X is a T_1 space if $\forall x, y \in X$, $x \neq y$ $\exists$ open sets U, V such that $x \in U$ and $x \notin V$, $y \in V$ and $y \notin U$.



<u>Proposition</u>: A space X is T_1 iff singletons are closed.

<u>Proof</u>: ($\Longrightarrow$) Let $x \in X$ be given, and choose $y \in \{x\}^c$. Since X is T_1, $\exists$ V open s.t. $y \in V \subset \{x\}^c$. Thus y is an interior point of $\{x\}^c$, so $\{x\}^c$ is open and $\{x\}$ is closed.

($\Longleftarrow$) Suppose all singletons are closed and let $x, y \in X$, $x \neq y$ be given. Since $\{x\}^c$ and $\{y\}^c$ are open, $\exists$ U, V open s.t. $x \in U \subset \{y\}^c$ and $y \in V \subset \{x\}^c$, which shows that X is T_1.

<u>Example</u>: The following space is T_0 but not T_1:

Fix any set X, and fix $x_0 \in X$. Define a topology

$$\mathcal{T} = \{\emptyset\} \cup \{U \subset X \mid x_0 \in U\}.$$

Then $\mathcal{T}$ is a topology.

We check that X is T_0:

Given $x, y \in X$ with $x \neq y$, suppose neither x nor y is x_0. Then set $U = \{x_0, x\}$ and $V = \{x_0, y\}$.

On the other hand if one of them is x_0, say x, then take $U = \{x_0\}$ and $V = \{x_0, y\}$.

This also illustrates why it is not T_1: If $x = x_0$, then every open set must contain x, so given $y \neq x$ $\nexists\, V$ open s.t. $y \in V$ and $x \notin V$.

Conclusion: It is obvious that every T_1 space is T_0, but not all T_0 spaces are T_1.

$$\{T_1 \text{ spaces}\} \underset{\text{proper}}{\subset} \{T_0 \text{ spaces}\}$$

Definition: A space is T_2 iff it is Hausdorff.

Example: Let X be an infinite set with the cofinite topology.

Then X is T_1 since $\forall x, y \in X$, we can take $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$. However, X is not Hausdorff because any two sets with finite complement must intersect, since X is infinite.

Thus $\{T_2 \text{ spaces}\} \underset{\text{proper}}{\subset} \{T_1 \text{ spaces}\}$

Definitions: Let A and B be disjoint subsets of a space X. We say that open sets $U, V \subset X$ separate A and B if $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

A space X is called regular if for every $x \in X$ and every closed $F \subset X$ $\exists$ U, V open that separate $\{x\}$ and F.

A space is called a T_3-space if: it is both regular and it is a T_1 space. Elsewhere in the literature you will find that X is a T_3 space if it is T_1 (there is some conflict here).

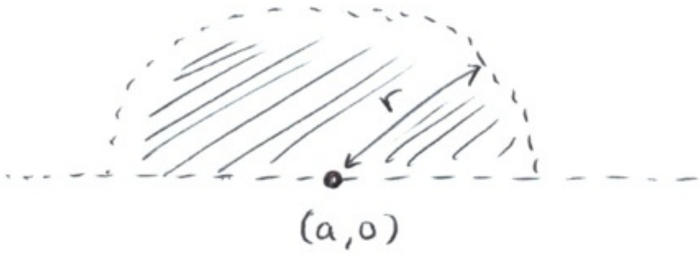
Example: Consider $X = \mathbb{R}^2_{up} = \{(x,y) \mid y \geq 0\}$. As a basis for the topology take all open discs contained in $\mathbb{R}^2_{up}$, together with sets of the form

$$\{(a,0)\} \cup \{(x,y) \mid (x-a)^2 + y^2 < r,\ y > 0\ r \in \mathbb{R}_{>0}\}.$$

i.e.

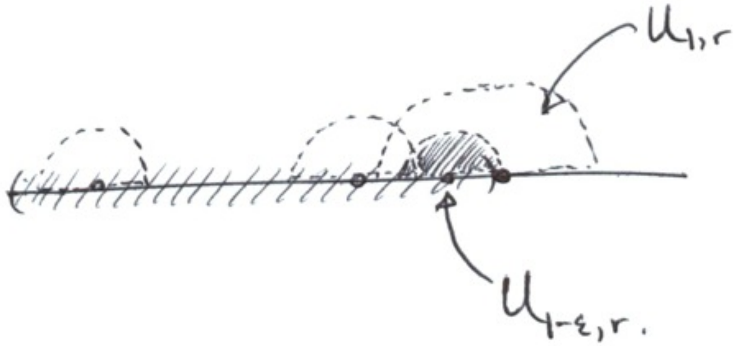
Then $\mathbb{R}^2_{up}$ is Hausdorff / T_2 in this topology since every pair of points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_{up}$ has a pair of disjoint open balls around them.

However, $\mathbb{R}^2_{up}$ is not regular, so not T_3. To see this, note that $\{(x,0) \mid x \in (0,1)\}$ is a closed subset of $\mathbb{R}^2_{up}$: If we write $U_{a,r}$ for the open set:



then $\{(x,0) \mid x \in (0,1)\}^c = \bigcup\limits_{\substack{a \notin (0,1) \\ r>0}} U_{a,r}$, a union of open sets. However, this closed set cannot be separated from the point $(1,0) \in \mathbb{R}^2_{up}$.

Any open set V containing $\{(x,0) \mid x \in (0,1)\}$ must contain a set $U_{a,r}$ $\forall a \in (0,1)$. Thus for any $r>0$, the open basic set $U_{1,r}$ must intersect a set of the form $U_{1-\varepsilon,r}$ for ε sufficiently small.



Thus not every T_2 space is a T_3 space, however

Proposition: Every T_3 space is a T_2 space.

(Note: This is not automatic unless we include T_1 in the definition of T_3 (as we have). Now it's "automatic")

Proof: Let $x, y \in X$ be given. Since X is T_3, it is also T_1, and thus $\{x\}$ is closed. Now since X is reglar there exist U, V separating $\{x\}$ and y. These sets show that X is Hausdorff.

Therefore

$$\{T_3 \text{ spaces}\} \underset{\text{proper.}}{\subset} \{T_2 \text{ spaces}\}$$

Def: A space is normal if for every pair of disjoint closed sets F, F', there exist open U, V that separate F and F'. A space is called a T_4 space if it is normal and is a T_1-space.

Fact: The containment

$$\{T_4 \text{ spaces}\} \subset \{T_3 \text{ spaces}\}$$

is also proper, but we have to wait for an example:

Theorem: If X is a T_i-space and $A \subset X$, then A is a T_i space.

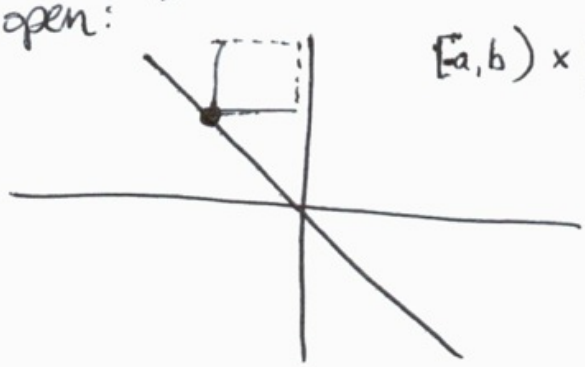
Proof: Definition of the subspace topology.

Theorem: If $\{X_i\}_{i \in I}$ are T_i spaces, so is $\prod_{i \in I} X_i$.

Proof: Deferred.

Example: The Sorgenfrey line is T_3. To see that it is T_1 is straight forward, so let F be a closed subset of $\mathbb{R}_s$ (Sorgenfrey topology) and choose $x \in \mathbb{R} \setminus F$. Then there exists $[x,b) \subset \mathbb{R} \setminus F$ since x is not an accumulation point of F. But $[x,b)^c = (-\infty, x) \cup [b, \infty)$ is open in $\mathbb{R}_s$, and so $[x,b)$ and $[x,b)^c$ separate x and F.

Now by the preceding theorem, $\mathbb{R}_s \times \mathbb{R}_s$ is T_3 as well. However it is not T_4:
Consider the subset of $\mathbb{R}_s \times \mathbb{R}_s$ defined by $y = -x$. The subspace topology on this set is discrete, as singletons are open:



$[-a,b) \times [a,c)$ shows $\{(-a,a)\}$ open

Thus the subsets:

$$A = \{(a,-a) \mid a \in \mathbb{Q}\}$$

and $B = \{(a,-a) \mid a \in \mathbb{R} \setminus \mathbb{Q}\}$

are closed subsets of $\{(a,-a) \mid a \in \mathbb{R}\}$.

By definition of the subspace topology there exist closed sets F_A and F_B in $\mathbb{R}_s \times \mathbb{R}_s$ such that

$$F_A \cap \{(a,-a) \mid a \in \mathbb{R}\} = A \ , \ F_B \cap \{(a,-a) \mid a \in \mathbb{R}\} = B.$$

The closed sets F_A and F_B cannot be separated by any open sets $\mathcal{U}$, V, any basic open set containing some $(a,-a) \in A$ must intersect B.

So we have

$$\underset{\substack{\text{normal} \\ +T_1}}{T_4} \subset \underset{\substack{\text{regular} \\ +T_1}}{T_3} \subset \underset{\text{Hausdorff}}{T_2} \subset T_1 \subset T_0$$

and all are strict containment.

MATH 3240 Topology. April 3.

Last day we ended with two theorems that we'll use to construct an example of a space which is T_3, but not T_4. Because I exceeded the limit of one definition/15 mins, let me remind you:

Def: A space is T_3 if (i) $\forall x \in X$ and closed $F \subset X$ with $x \notin F$, $\exists$ U, V separating x and F (regular), and (ii) it is also T_1.

Def: A space is T_4 if (i) $\forall$ pairs of disjoint closed sets $F, F' \subset X$ there exist U, V separating F and F', (normal), and (ii) it is also T_1.

Theorem: If $\{X_j\}_{j \in J}$ are T_i spaces ($i \neq 4$) then so is $\prod_{j \in J} X_j$.

Proof: We do the case of T_3 spaces.

First we show the product is T_1:

Let $(x_j), (y_j) \in \prod_{j \in J} X_j$ be given, and fix some $j_0 \in J$. Then since X_{j_0} is T_1, $\exists$ open $U, V \subseteq X_{j_0}$ such that $x_{j_0} \in U \setminus V$, $y_{j_0} \in V \setminus U$.

Therefore the open sets

$$U \times \prod_{j \in J \setminus \{j_0\}} X_j \;, \quad V \times \prod_{j \in J \setminus \{j_0\}} X_j$$

show that the product is T_1 (note it's ok if the open sets overlap).

Now we show the product is regular, so fix $(x_j) \in \prod_{j \in J} X_j$ and a closed set $F \subset \prod_{j \in J} X_j$ not containing (x_j). Then $(x_j) \in F^c$, an open set, so there is a standard basic set $U = \prod_{j \in J} U_j$ such that $(x_j) \in \prod_{j \in J} U_j \subseteq F^c$.

As each U_j^c is closed and $x_j \notin U_j^c$, we can use regularity of X_j to choose an open set V_j and W_j with $x_j \in V_j$, $U_j^c \subset W_j$ and $V_j \cap W_j = \emptyset$. (Note if $U_j = X_j$ we simply take $V_j = W_j = X_j$). Then $\prod_{j \in J} V_j$ is a basic open set containing (x_j), $\prod_{j \in J} W_j$ is a basic open set containing F, and by construction their intersection is empty.

Now we are prepare to show

$$\{T_4 \text{ spaces}\} \subset \{T_3 \text{ spaces}\}$$

proper.

(skip to last day's notes)

§8.2 : Regularity and Normality.

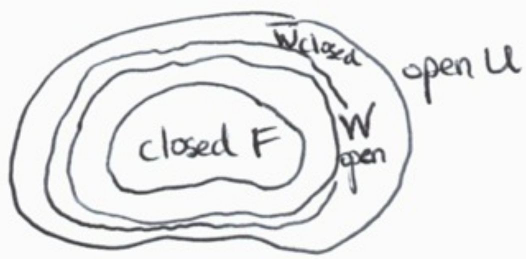
Because normality fails to obey the same properties as regularity, we wish to investigate their difference carefully. What do we need to add to regularity to get normality? First,

Theorem: For a space X, the following are equivalent:

(i) X is normal

(ii) For every closed $F \subset X$ and every open $\mathcal{U}$ such that $F \subset \mathcal{U}$, there is an open W such that

$$F \subset W \subset \overline{W} \subset \mathcal{U}$$



(iii) For every two disjoint closed $F, G \subset X$ $\exists$ $\mathcal{U}, V$ open st. $F \subset \mathcal{U}$, $G \subset V$ and $\overline{\mathcal{U}} \cap \overline{V} = \emptyset$.

Proof: (i) $\Rightarrow$ (ii).

Choose a closed F and open $\mathcal{U}$ s.t. $F \subset \mathcal{U}$. Then F and $\mathcal{U}^c$ are disjoint closed sets, by normality $\exists$ open V, W s.t. $F \subset W$ and $\mathcal{U}^c \subset V$.

Since $W \cap V = \emptyset$, $W \subset V^c$. But V^c is closed so $\overline{W} \subset \overline{V^c} = V^c$, and $V^c \subset \mathcal{U}$ since $\mathcal{U}^c \subset V$. Overall,

$$F \subset W \subset \overline{W} \subset \overline{V^c} = V^c \subset \mathcal{U},$$ and we have the W we wanted.

(ii) $\Rightarrow$ (iii)

Let $F, G \subset X$ be disjoint and closed. Then G^c is open and $F \subset G^c$, so by (ii) $\exists\, U$ s.t. $F \subset U \subset \overline{U} \subset G^c$. Then $\overline{U} \subset G^c$ gives $G \subset (\overline{U})^c$ and since G is closed and $(\overline{U})^c$ is open, we apply (ii) again to find V open s.t. $G \subset V \subset \overline{V} \subset (\overline{U})^c$. Then $F \subset U$, $G \subset V$, and since $\overline{V} \subset (\overline{U})^c$, $\overline{V} \cap \overline{U} = \emptyset$, as required.

(iii) $\Rightarrow$ (i). This is obvious, since we're given (by (iii)) open sets U, V with $\overline{U} \cap \overline{V} = \emptyset$, and all we required for normality was $U \cap V = \emptyset$.

<u>Theorem</u>: A space is regular iff for every $x \in X$, and for every open U with $x \in U$, $\exists\, W$ open s.t. $x \in W \subset \overline{W} \subset U$.

<u>Proof</u>: Pretty much identical to the proof of (ii) in the previous theorem, replacing one of the closed sets with the singleton $\{x\}$. (Recall $\{x\}$ is closed since T_3 for us also includes T_1, so in T_3 situations the proof actually <u>is</u> identical).

Now we are ready to show what we can add to regularity to get normality.

__Theorem__: A regular Lindelöf space is normal.

__Proof__: Suppose X is regular and Lindelöf, and let A, B be disjoint closed subsets of X. Since X is regular, for every $a \in A$ $\exists$ U_a such that $a \in \overline{U_a} \subset B^c$ (U_a open). Since A is a subspace of a Lindelöf space it too is Lindelöf, so from the open cover $\{U_a \cap A\}$ of A we extract a countable subcover $\{U_{a_i} \cap A\}_{i=1}^{\infty}$. For simplicity we rename U_{a_i} to U_i, and consider the subsets $\{U_i\}_{i=1}^{\infty}$ of X, which satisfy

$$A \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \overline{U_i} \subset B^c \ \forall i \text{ by construction.}$$

Similarly construct open V_i with

$$B \subset \bigcup_{i=1}^{\infty} V_i \text{ and } \overline{V_i} \subset A^c \ \forall i \text{ by construction.}$$

Define two new sequences of open sets:

$$U_n' = U_n \setminus \left(\bigcup_{i=1}^{n} \overline{V_i}\right)$$

$$V_n' = V_n \setminus \left(\bigcup_{i=1}^{n} \overline{U_i}\right).$$

We need to check that $U_n' \cap V_m' = \emptyset$ $\forall n, m$, and then $U = \bigcup_{n=1}^{\infty} U_n'$, $V = \bigcup_{n=1}^{\infty} V_n'$ will be the open sets required for normality.