MATH 3240 Topology 1.

Last day we ended with a generalization of compactness:

A space X is <u>Lindelöf</u> if every open cover has a countable subcover.

<u>Example</u>: The Sorgenfrey line is not compact, but is Lindelöf. Note it is <u>not</u> second countable, because...

<u>Proposition</u>: (Lindelöf lemma) Every second countable space is Lindelöf.

<u>Proof</u>: Let X be a second countable space and $\mathcal{W}$ an open cover of X. Let $\mathcal{B}$ be a countable basis of X. Each $W \in \mathcal{W}$ is a union of elements of $\mathcal{B}$, so create a new countable cover $\mathcal{V}$ of X that consists of all $B \in \mathcal{B}$ that are used in writing some $W \in \mathcal{W}$ as a union of basic elements. I.e, if

$$W = \bigcup_{i \in I} B_i \text{ , then } \{B_i\}_{i \in I} \subset \mathcal{V}.$$

Now choose a countable subcover $\mathcal{W}' \subset \mathcal{W}$ as follows: for each $V \in \mathcal{V}$, choose $W \in \mathcal{W}$ s.t. $V \subset W$. Then the collection of all such W's is $\mathcal{W}'$, which is countable.

<u>Example</u>: This shows that $\mathbb{R}$ is Lindelöf, but is not compact.

Recall the Bolzano Weierstrass theorem:

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Inspired by this, we define a Bolzano-Weierstrass space (or BW space), to be a space X in which every infinite subset has an accumulation point.

First, we note:

Def: x is an accumulation point of A if every open subset containing x contains a point of A other than x.

Lemma: Suppose X is Hausdorff and $A \subseteq X$. If $p \in X$ is an accumulation point of A, then every neighbourhood of p contains infinitely many points of A.

Proof: Construct infinitely many points $\{x_n\}$ as follows, given a nbhd U of p:

$\exists\, x_1 \in U \cap A$ since p is an accumulation point. Now suppose we have $x_1, \dots, x_k$. For each x_i, $\exists$ nbhds U_i of x_i and V_i of p such that $U_i \cap V_i = \emptyset$. Set

$$V = \left(\bigcap_{i=1}^{k} V_i\right) \cap U,$$ which is an open neighbourhood of p and so contains a point x_{k+1} of A and not any of $x_1, \dots, x_k$. //

Proposition: Suppose that X is a Hausdorff space. Then X is a BW-space iff it is countably compact.

Proof: ($\Rightarrow$) Suppose X is a BW-space. Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be a countable open covering of X, assume that no U_i is contained in $U_1 \cup U_2 \cup \ldots \cup U_{i-1}$ (eliminate redundancy). Suppose $\mathcal{U}$ has no finite subcovering.

Then there's a set

$$A = \{x_n \in X \mid x_n \in U_n \setminus (\bigcup_{i=1}^{n-1} U_i)\},$$

and the set A is infinite since $x_i \neq x_j$ if $i \neq j$. Since X is a BW space, A has an accumulation point x. Since $\mathcal{U}$ is a cover, there exists n such that $x \in U_n$, and therefore U_n contains infinitely many points of A (here use the lemma). However for $m > n$, $x_m \notin U_n$ by construction, a contradiction.

($\Leftarrow$) Suppose that X is countably compact. We show that every countably infinite subset has an accumulation point. Suppose $A = \{a_1, a_2, \ldots\}$ does not have an accumulation point. Then A is closed since $\bar{A} = A \cup A' = A$, and since each $a_i \in A$ is not an accumulation point of A $\exists$ an open nbhd U_i of a_i s.t. $U_i \cap A = \{a_i\}$. Thus $\{U_i\}_{i=1}^{\infty} \cup \{(X \setminus A)\}$ is an open covering of X, so it must have a finite subcovering, say $\{U_1, \ldots, U_n\} \cup \{X \setminus A\}$.

But then some U_i must contain infinitely many of the a_i's, a contradiction.

Proposition: Every compact space is a BW space.

Proof: Suppose X is not a BW space. Then there is an infinite subset of A without accumulation points in X. Thus, A contains all its accumulation points and so is closed.

Now if X is compact, then A is compact since it is closed. Moreover since A has no accumulation points, for each $a \in A$ $\exists$ U_a an open nbhd of a such that $U_a \cap A = \{a\}$. Then $\{U_a\}_{a \in A}$ is an open cover of A, so we choose a finite subcover $\{U_{a_1}, \dots, U_{a_n}\}$. Then

$$A = \left(\bigcup_{i=1}^{n} U_{a_i}\right) \cap A = \bigcup_{i=1}^{n} (A \cap U_{a_i}) = \{a_1, \dots, a_n\},$$ so that

A is finite, a contradiction.

Our goal now is to show that for metric spaces, the converse also holds: If (X, d) is a Bolzano-W. space, then X is compact. This requires a famous lemma

Lemma ~~(Lesbegu~~ (Lebesgue number lemma)

~~For every~~ Let (X, d) be a BW-metric space, and suppose that $\mathcal{W}$ is an open cover of X.

Then $\exists\, \varepsilon>0$ s.t. $\forall x\in X\ \exists\, W\in\mathcal{W}$ with $B(x,\varepsilon)\subset W$.
(i.e. There's a radius $\varepsilon>0$ called the Lebesgue number such that every ε-ball is contained in some element of the cover).

Proof: Let X be a BW-space with metric d, $\mathcal{W}$ an open cover, and assume that $\exists\, x\in X$ s.t. $\forall \varepsilon>0$ the ball $B(x,\varepsilon)$ is not a subset of any $U\in\mathcal{W}$. In particular, for every $\varepsilon=\frac{1}{n}$, $n\in\mathbb{N}^+$, there is a point $x_n\in X$ such that $B(x_n,\frac{1}{n})$ is not contained in any $U\in\mathcal{W}$.
First, note that $\{x_i\}_{i=1}^{\infty}$ is an infinite set. If not, then $x_m=x_n$ $\forall m\geq n$ for some n, and the statement "$B(\overset{x_n}{\overset{\parallel}{x_m}},\frac{1}{m})$ is not a member of any $U\in\mathcal{W}$ $\forall m\geq n$" contradicts the fact that the balls $\{B(x_n,\frac{1}{m})\}_{m\in\mathbb{N}^+}$ form a local basis.

So, since X is a BW-space the sequence $\{x_i\}_{i=1}^{\infty}$ has an accumulation point, say $x\in X$. Choose $U\in\mathcal{W}$ containing x, and a ball $B(x,r)\subset U$. Since X is ~~Choose~~ a metric space, $B(x,\frac{r}{2})$ contains infinitely many of the points $\{x_i\}_{i=1}^{\infty}$.
Choose $m\in\mathbb{N}$ such that $\frac{1}{m}<\frac{r}{2}$, such that $x_m\in B(x,\frac{r}{2})$.
Then $B(x_m,\frac{1}{m})\subset B(x,r)\subset U\in\mathcal{W}$, contradicting our choice of x_m.

MATH 3240 Topology 1.

Recall:

Zorn's lemma: Let P be a partially ordered set. If every totally ordered subset of P has an upper bound, then P contains a maximal element.

I.e. $\exists x \in P$ st. if x is comparable to $y \in P$, then $y < x$.

We'll use this to prove

Theorem (Alexander subbase theorem). Let X be a space with topology $\mathcal{T}$ having subbasis S. If every collection of set from S that covers X has a finite subcover, then X is compact.

~~Lemma~~: Proof: By contradiction. Suppose that every cover by elements of S has a finite subcover, yet X is not compact.

$$\text{Let } \mathcal{F} = \{\mathcal{W} \mid \mathcal{W} \text{ is an open cover of } X \text{ with no finite subcover}\}.$$

Then $\mathcal{F}$ is nonempty, and it is partially ordered by inclusion (say $\mathcal{W}_1 < \mathcal{W}_2$ if both are in $\mathcal{F}$ and $\mathcal{W}_1 \subset \mathcal{W}_2$).

We would like to apply Zorn's lemma, so let $\{\mathcal{W}_i\}_{i \in I}$ be a totally ordered subset of $\mathcal{F}$. Then

Claim: $\mathcal{W} = \bigcup_{i \in I} \mathcal{W}_i$ is an upper bound for $\{\mathcal{W}_i\}_{i \in I}$.

Certainly we have $\mathcal{W}_i \subset \mathcal{W}$ $\forall i$, so we only need to check that $\mathcal{W}$ is an open cover with no finite subcover.

Obviously $\mathcal{W}$ covers X, so suppose $W_1, \dots, W_n$ is a finite subcover. Each W_k is contained in some cover $\mathcal{W}_{i_k}$, and since $\mathcal{W}_{i_1}, \mathcal{W}_{i_2}, \dots, \mathcal{W}_{i_n}$ are totally ordered by inclusion, there exists a largest $\mathcal{W}_{i_k}$. Then $W_1, \dots, W_n \in \mathcal{W}_{i_k}$ and $W_1, \dots, W_n$ is a finite subcover of $\mathcal{W}_{i_k}$, a contradiction.

Thus, Zorn's Lemma gives us a maximal element of $\mathcal{F}$, call it $\mathcal{U}$. Set

$$S' = \mathcal{U} \cap S,$$

then our next claim is:

<u>Claim</u>: S' covers X.

If not, choose $x \notin \bigcup_{B \in S'} B$. Then $\exists\, U \in \mathcal{U}$ with $x \in U$, since $\mathcal{U}$ covers X. Since S is a subbasis and U is open, $\exists\, V_1, \dots, V_n \in S$ s.t. $x \in \bigcap_{i=1}^{n} V_i \subset U$, moreover $V_i \notin \mathcal{U}\ \forall i$ since then x would be in $\bigcup_{B \in S'} B$.

But then by maximality of $\mathcal{U}$, every cover $\mathcal{U} \cup \{V_i\}$ must contain a finite subcover. Say $X = U_i \cup V_i$, where U_i is the union of all sets in the finite subcover of $\mathcal{U} \cup \{V_i\}$ (except V_i). Then

$$U \cup \left(\bigcup_{i=1}^{n} U_i\right) \supseteq \underbrace{\left(\bigcap_{i=1}^{n} V_i\right) \cup \left(\bigcup_{i=1}^{n} U_i\right) \supseteq \bigcap_{i=1}^{n} (V_i \cup U_i)} = X.$$

This is $\left(\bigcap_{i=1}^{n} V_i\right) \cup \left(\bigcup_{j=1}^{n} U_j\right) = \bigcap_{i=1}^{n}\left(V_i \cup \bigcup_{j=1}^{n} U_j\right)$

replace with $V_i \cup U_i$ since already $V_i \cup U_i = X$

Which is impossible, since $\mathcal{U}$ is not supposed to have any finite subcovers.

Therefore, $S' = S \cap \mathcal{U}$ is a cover of X. But then S' has a finite subcover since $S' \subset S$ (by assumption). But since $S' \subset \mathcal{U}$ this would mean $\mathcal{U}$ has a finite subcover. Thus our collection $\mathcal{F}$ must be empty in order to avoid this contradiction. Thus, X is compact.

Theorem (Tychonoff). If (X_i, τ_i) are compact spaces $\forall i \in I$, then $\prod_{i \in I} X_i$ is also compact.

Proof:

Lemma: Any open cover of $\prod_{i \in I} X_i$ containing only elements of the form $p_i^{-1}(U)$ $(U \in \tau_i)$ admits a finite subcover.

Proof of Lemma: Let $\mathcal{U}$ be such a cover, and set

$$\mathcal{U}_i = \{ U \in \tau_i \mid p_i^{-1}(U) \in \mathcal{U} \}.$$

Claim: There's at least one $i \in I$ such that $\mathcal{U}_i$ is a cover of X_i. If not, then for each i $\exists x_i \in X_i$ s.t. $x_i \notin \bigcup_{U \in \mathcal{U}_i} U$; consider $(x_i) \in \prod_{i \in I} X_i$. The point (x_i) cannot be contained in any set $p_i^{-1}(U)$ by construction, this contradicts the fact that $\mathcal{U}$ is a cover of $\prod_{i \in I} X_i$.

So by our claim, we can choose $i \in I$ such that $\mathcal{U}_i$ is a cover of X_i. But X_i is compact, so $\exists$ $V_1, \dots, V_n \in \mathcal{U}_i$ s.t. $X_i \subset \bigcup_{j=1}^{n} V_j$. But then $\{p_i^{-1}(V_1), p_i^{-1}(V_2), \dots, p_i^{-1}(V_n)\}$ is a finite subcover of $\mathcal{U}$, and it covers $\prod_{i \in I} X_i$. The lemma is proved.

<u>Proof of Tychonoff's theorem:</u>

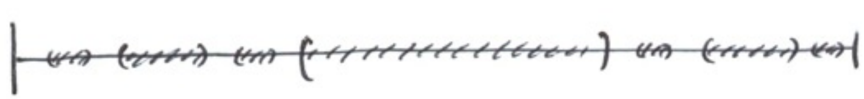
Recall that a subbasis for the product topology is

$$S = \{p_i^{-1}(U) \mid U \in \mathcal{T}_i, i \in I\}.$$

By our lemma, any collection of sets of this form must have a finite subcover if it covers X. By Alexander's subbasis theorem, $\prod_{i \in I} X_i$ is compact.

=== DONE! ===

<u>Example</u>: Recall the middle-thirds construction of the Cantor set, C:



the remaining points after these successive deletions form C.

We saw by a clever base 3 expansion argument that there is a homeomorphism

$$h : C \longrightarrow \prod_{i=1}^{\infty} \{0,2\},$$

where $\{0,2\}$ has the discrete topology. Note $\{0,2\}$ is compact, and thus by Tychonoff's theorem, the Cantor set C is compact.

<u>Example</u>: The power set of a set X can be identified with $\prod_{x \in X} \{0,1\}$, a sequence (y_x) corresponds to a subset A of X as follows:

$$y_x = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then the power set $P(X) = \prod_{x \in X} \{0,1\}$ inherits a "natural" topology which makes it a compact topological space.

<u>Applications mentioned in text</u>:

- The space $\mathbb{R}^\infty$ is metrizable.
- Ramsey theory?

Sketch of content to come:

We define a bunch of properties that are variations of Hausdorff-ness, the are named as follows:

T_0

T_1

T_2 = Hausdorff

T_3

T_4

normal ← variations of this are 4-6, usually.

regular.

Complete list:

$T_0, T_1, T_2, T_{2\frac{1}{2}}$, completely T_2, $T_3, T_{3\frac{1}{2}}, T_4, T_5, T_6$

We'll give many examples of spaces that have one property, but not the other. "Separation axioms"

Definition: A space X is a T_0-space if for every $x, y \in X$ there are open sets U, V such that $x \in U$, $y \in V$ and either $x \notin U$ or $y \notin V$.

Or equivalently: For all $x, y \in X$ $\exists$ U open that contains exactly one of x, y.

Example: • The trivial topology $\{\emptyset, X\}$ on any set X with more than two points is not T_0.

• Give $\mathbb{R}_t$ the trivial topology, $\mathbb{R}$ the usual topology. Then $\mathbb{R}_t \times \mathbb{R}$ with the product topology is not T_0, points (a, b) and (c, b) fail the condition